FREQUENCY RESPONSE ANALYSIS

Introduction

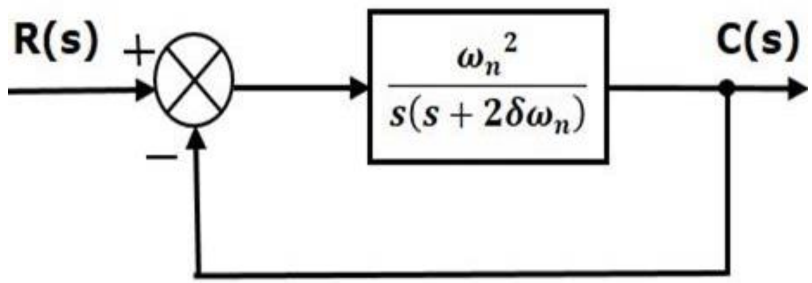
The steady state response of a system for an input sinusoidal signal is known as the **frequency response**

Consider a linear system with a sinusoidal input

$$r(t) = A \sin \omega t$$

- Under steady-state, the system output as well as the signals at all other points in the system are sinusoidal.
- The steady-state output is $c(t) = B \sin(\omega t + \emptyset)$
- The magnitude and phase relationship between the sinusoidal input and the steady-state of a system is termed as Frequency Response.

- In LTI systems, the frequency response is independent of amplitude and phase of the input signal.
- The analysis in frequency domain is easy than in time domain.
- The frequency response is evaluated from the sinusoidal transfer function by replacing s by $j\omega$ in the system transfer function $T(s)$
- The transfer function $T(j\omega)$ has both magnitude and phase angle.
- The characteristics are conveniently represented by graphical plots.



Consider the transfer function of the second order closed control system as

$$T(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

Substitute, $s = j\omega$ in the above equation.

$$T(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\delta\omega_n(j\omega) + \omega_n^2}$$

$$\Rightarrow T(j\omega) = \frac{\omega_n^2}{-\omega^2 + 2j\delta\omega x\omega_n + \omega_n^2} = \frac{\omega_n^2}{\omega_n^2\left(1 - \frac{\omega^2}{\omega_n^2} + \frac{2j\delta\omega}{\omega_n}\right)}$$

$$\Rightarrow T(j\omega) = \frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right) + j\left(\frac{2\delta\omega}{\omega_n}\right)}$$

Let, $\frac{\omega}{\omega_n} = u$ Substitute this value in the above equation.

$$T(j\omega) = \frac{1}{(1 - u^2) + j(2\delta u)}$$

Magnitude of $T(j\omega)$ is -

$$M = |T(j\omega)| = \frac{1}{\sqrt{(1-u^2)^2 + (2\delta u)^2}}$$

Phase of $T(j\omega)$ is -

$$\varnothing = \angle T(j\omega) = -tan^{-1}\left(\frac{2\delta u}{1-u^2}\right)$$

The loop transfer function of a system is given by G(s)= $\frac{100(s+2)}{s(s+1)(s+5)}$

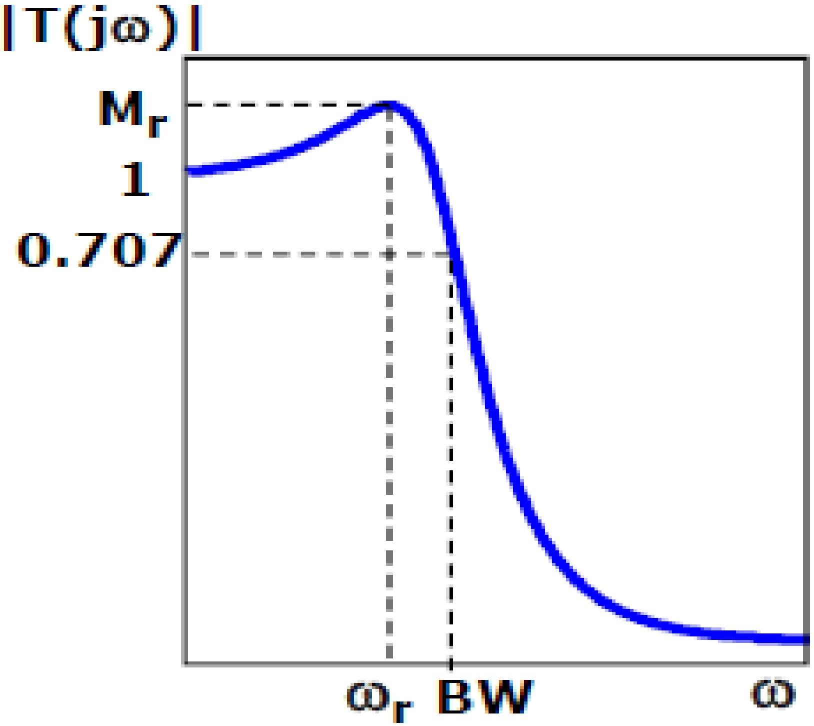
Find the magnitude and phase at 10 rad/sec.

M= A+jB = $\sqrt{A^2+B^2}$

Ø = $\tan^{-1} B/A$

Frequency Domain Specifications

- Resonant peak
- Resonant frequency
- Bandwidth.



Resonant Frequency

Resonant frequency, ωr :

This is the frequency at which the resonant peak is obtained.

Differentiate M with respect to u.

$$\frac{\mathrm{d}M}{\mathrm{d}u} = -\frac{1}{2}\left[(1-u^2)^2 + (2\delta u)^2\right]^{\frac{-3}{2}}\left[2(1-u^2)(-2u) + 2(2\delta u)(2\delta)\right]$$

$$\Rightarrow \frac{\mathrm{d}M}{\mathrm{d}u} = -\frac{1}{2}\left[(1-u^2)^2 + (2\delta u)^2\right]^{\frac{-3}{2}}\left[4u(u^2 - 1 + 2\delta^2)\right]$$

Substitute, $u = u_r$ and $\frac{\mathrm{d}M}{\mathrm{d}u} == 0$ in the above equation.

$$0 = -\frac{1}{2}\left[(1-u_r^2)^2 + (2\delta u_r)^2\right]^{-\frac{3}{2}}\left[4u_r(u_r^2 - 1 + 2\delta^2)\right]$$

$$\Rightarrow 4u_r(u_r^2 - 1 + 2\delta^2) = 0$$

$$\Rightarrow u_r^2 - 1 + 2\delta^2 = 0$$

$$\Rightarrow u_r^2 = 1 - 2\delta^2$$

$$\Rightarrow u_r = \sqrt{1 - 2\delta^2}$$

Substitute, $u_r = \frac{\omega_r}{\omega_n}$ in the above equation.

$$\frac{\omega_r}{\omega_n} = \sqrt{1 - 2\delta^2}$$

$$\Rightarrow \omega_r = \omega_n \sqrt{1 - 2\delta^2}$$

Resonant Peak

Resonant Peak

It is the peak (maximum) value of the magnitude of T(jω). It is denoted by M_r.

At u=u_r, the Magnitude of T(jω) is -

$$M_r = \frac{1}{\sqrt{(1-u_r^2)^2 + (2\delta u_r)^2}}$$

Substitute, $u_r = \sqrt{1-2\delta^2}$ and $1-u_r^2 = 2\delta^2$ in the above equation.

$$M_r = \frac{1}{\sqrt{(2\delta^2)^2 + (2\delta\sqrt{1-2\delta^2})^2}}$$

$$\Rightarrow M_r = \frac{1}{2\delta\sqrt{1-\delta^2}}$$

Bandwidth

Bandwidth

It is the range of frequencies over which, the magnitude of T(jω) drops to 70.7% from its zero frequency value.

At ω=0, the value of u will be zero.

Substitute, u=0 in M.

$$M = \frac{1}{\sqrt{(1-0^2)^2 + (2\delta(0))^2}} = 1$$

Therefore, the magnitude of T(jω) is one at ω=0

At 3-dB frequency, the magnitude of T(jω) will be 70.7% of magnitude of T(jω)) at ω=0

i.e., at $\omega = \omega_B, M = 0.707(1) = \frac{1}{\sqrt{2}}$

$$\Rightarrow M = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{(1-u_b^2)^2 + (2\delta u_b)^2}}$$

$$\Rightarrow 2 = (1-u_b^2)^2 + (2\delta)^2 u_b^2$$

Let, $u_b^2 = x$

$$\Rightarrow 2 = (1-x)^2 + (2\delta)^2 x$$

$$\Rightarrow x^2 + (4\delta^2 - 2)x - 1 = 0$$

$$\Rightarrow x = \frac{-(4\delta^2 - 2) \pm \sqrt{(4\delta^2 - 2)^2 + 4}}{2}$$

Consider only the positive value of x.

$$x = 1 - 2\delta^2 + \sqrt{(2\delta^2 - 1)^2 + 1}$$

$$\Rightarrow x = 1 - 2\delta^2 + \sqrt{(2 - 4\delta^2 + 4\delta^4)}$$

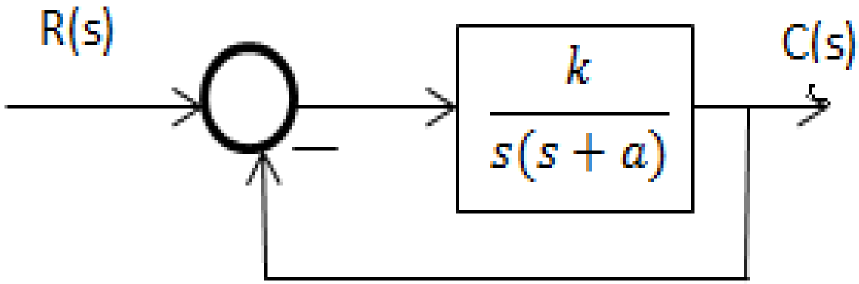
Substitute, $x = u_b^2 = \frac{\omega_b^2}{\omega_n^2}$

$$\frac{\omega_b^2}{\omega_n^2} = 1 - 2\delta^2 + \sqrt{(2 - 4\delta^2 + 4\delta^4)}$$

$$\Rightarrow \omega_b = \omega_n \sqrt{1 - 2\delta^2 + \sqrt{(2 - 4\delta^2 + 4\delta^4)}}$$

Problem 1

Consider the system as shown in figure,



(a) Find the value 'k' and 'a' to satisfy the following frequency domain specifications:

M_r = 1.04

ω_r = 11.55 rad/sec

(b) For this value of k and a , calculate settling time and bandwidth of the system

Solution

$$\frac{C(s)}{R(s)} = \frac{k}{s^2+as+k};$$

$$\omega_n{}^2 = k \qquad 2\zeta\omega_n = a$$

$M_r = 1.04$

$\omega_r = 11.55$ rad/sec

$$M_r = \frac{1}{2\sqrt{(1-2\zeta^2)}}$$

$$\omega_r = \omega_n\sqrt{(1-2\zeta^2)}$$

$\zeta = 0.6, 0.8$

$\omega_n = 21.8$ rad/sec

$k = 476$

$a = 26$

Problem 2

Unit- step response data of a second-order system is given below. Obtain the corresponding frequency response specifications for the system

t (sec)	0	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
C(t)	0	0.25	0.8	1.08	1.12	1.02	0.98	0.98	1.0	1.0	1.0

Solution

From the table, $t_p = 0.2$

$M_p = 0.12$

Find ζ and ω_n

Determine M_r, ω_r and BW.

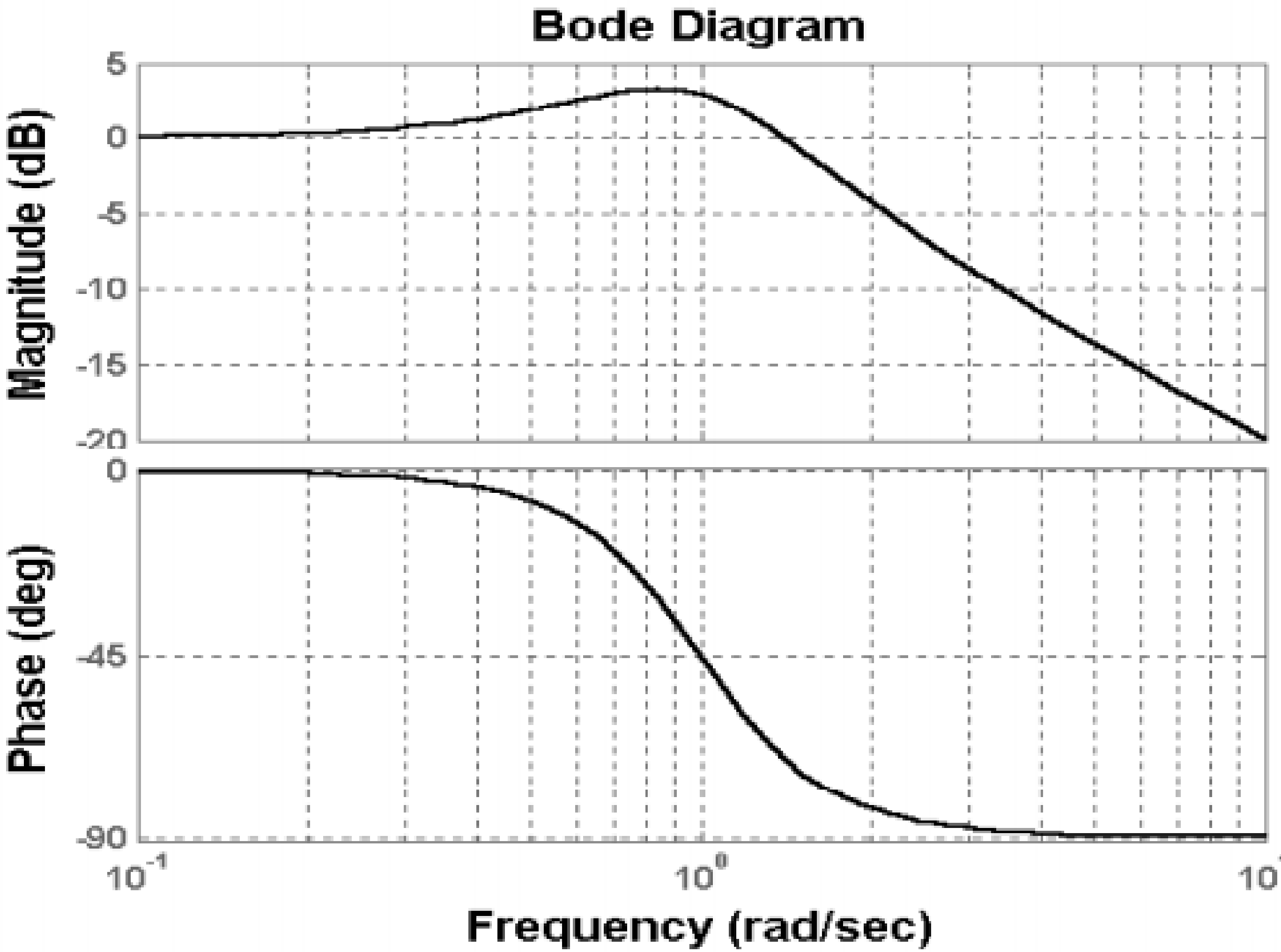
Frequency Domain Plots

1. Bode plot or Logarithmic plot
2. Nyquist plot

BODE PLOT

- The Bode plot of a transfer function is a useful graphical tool for the analysis and design of linear control systems in the frequency domain
- The Bode plot consists of two plots drawn on semi-logarithmic paper.

  1. Magnitude of the frequency response in decibels, i.e., 20 log| (G(jω)| on a linear scale versus frequency on a logarithmic scale.
  2. Phase of the frequency response function on a linear scale versus frequency on a logarithmic scale.



Bode Diagram
Magnitude (dB)
5
0
-5
-10
-15
-20
Phase (deg)
0
-45
-90
10^{-1}
10^{0}
10^{1}
Frequency (rad/sec)

Basic factors

Consider the following general transfer function

$$G(s) = \frac{k(1+Tas)(1+Tbs)\ldots\ldots}{s^r(1+T_1 s)(1+T_2 s)\ldots\ldots(s^2+2\zeta\omega_n s+\omega_n{}^2)}$$

$$G(j\omega) = \frac{k(1+j\omega T_a)(1+j\omega T_b)\ldots\ldots}{(j\omega)^r(1+j\omega T_1)(1+j\omega T_2)\ldots\left[1+j2\zeta\left(\frac{\omega}{\omega_n}\right)+\left(\frac{j\omega}{\omega_n}\right)^2\right]\ldots}$$

1. Gain k (Constant Term)
2. Integral or Derivative factors $(j\omega)^{\pm 1}$
   (Poles or zeros at origin)
3. First-order factors $(1+j\omega T)^{\pm 1}$
   (poles or zeros not at origin)
4. Quadratic factors $[1+2\zeta(j\omega/\omega_n)+(j\omega/\omega_n)^2]^{\pm 1}$
   (Complex poles or Complex zeros)

The magnitude of G(jω) = |G(jω)|

$$20\log|G(j\omega)| = 20\log|k| + 20\log|1+j\omega T_a| + 20\log|1+j\omega T_b| -$$
$$20r\log\omega - 20\log|1+j\omega T_1| - 20\log|1+j\omega T_2| \ldots -$$
$$20\log\left|1 + j2\zeta\left(\frac{\omega}{\omega_n}\right) - \left(\frac{\omega}{\omega_n}\right)^2\right| \ldots$$

The Phase of G(jω) = ∠G(jω)

$$\angle G(j\omega) = \tan^{-1}\omega T_a + \tan^{-1}\omega T_b + \ldots - r(90^o) - \tan^{-1}\omega T_1 -$$
$$\tan^{-1}\omega T_2 - \ldots - \tan^{-1}\left\{\frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)}\right\} \ldots$$

Gain K, Constant Term

- The gain factor multiplies the overall gain by a constant value for all frequencies.
- It has no effect on phase.

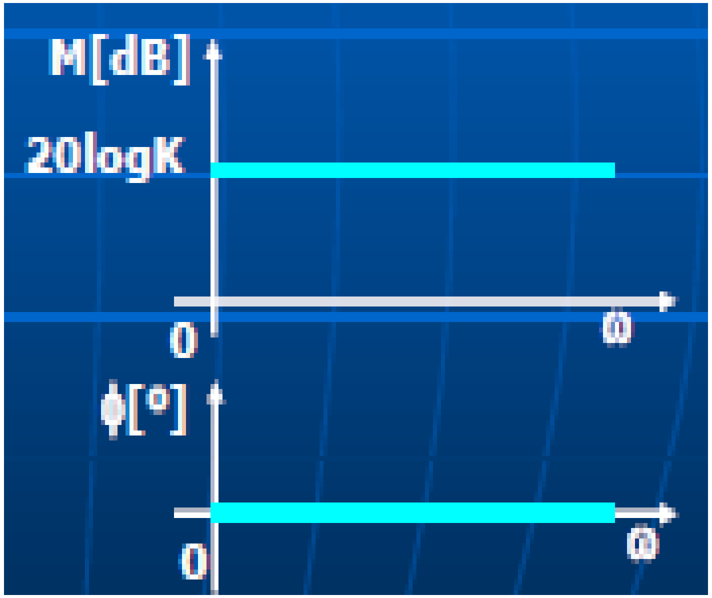
$$G(s)=K$$

$$G(j\omega)=k$$

$$M = 20\log |G(j\omega)|$$

$$M = 20\log K = \text{constant}$$

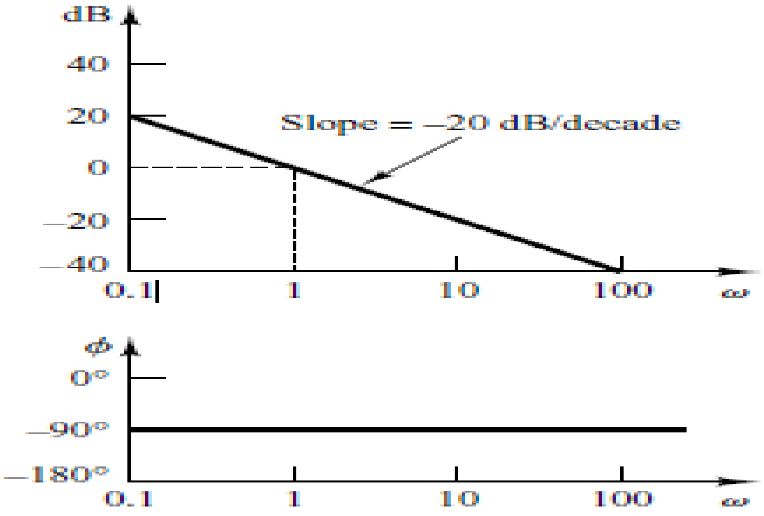
$$\emptyset = \angle G(j\omega) = \tan^{-1}\frac{0}{k} = 0^{o}$$



Integral Factor 1/jω – pole at origin

$$G(s) = \frac{1}{s}$$

$$G(j\omega) = \frac{1}{j\omega}$$



$$M = 20\log\left|\frac{1}{j\omega}\right| = -20\log \omega$$

Magnitude is a straight line with a slope of -20 dB/decade

$$\emptyset = -\tan^{-1}\frac{\omega}{0} = -90^{o}$$

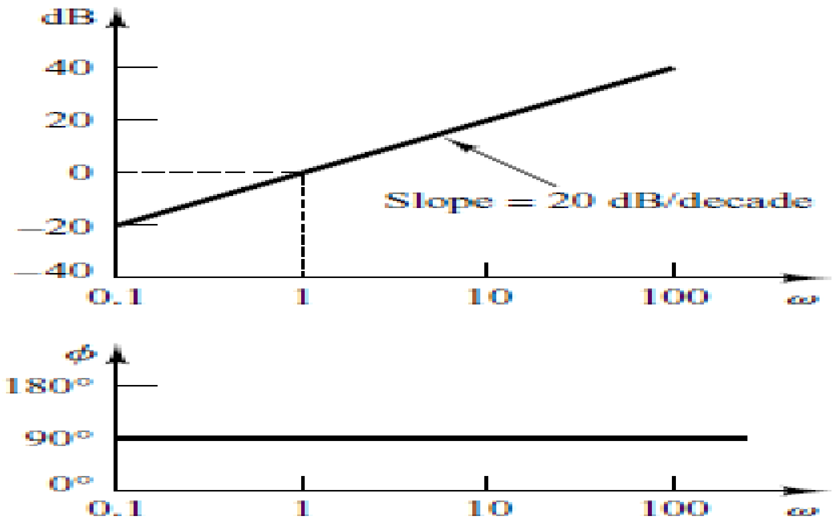
Phase is constant at -90° at all frequencies.

Derivative Factor jω – Zero at origin

$G(s) = s$

$G(j\omega) = j\omega$

$M = 20\log \omega$



Magnitude is a straight line with a slope of 20 dB/decade

$$\emptyset = \tan^{-1}\frac{\omega}{0} = 90^o$$

Phase is constant at 90º at all frequencies

Poles not at origin

$$G(s) = \frac{1}{1+Ts}$$

$$G(j\omega) = \frac{1}{1+Tj\omega}$$

$$M = 20\log \left| \frac{1}{1+Tj\omega} \right| \qquad \emptyset = -\tan^{-1}\omega T$$

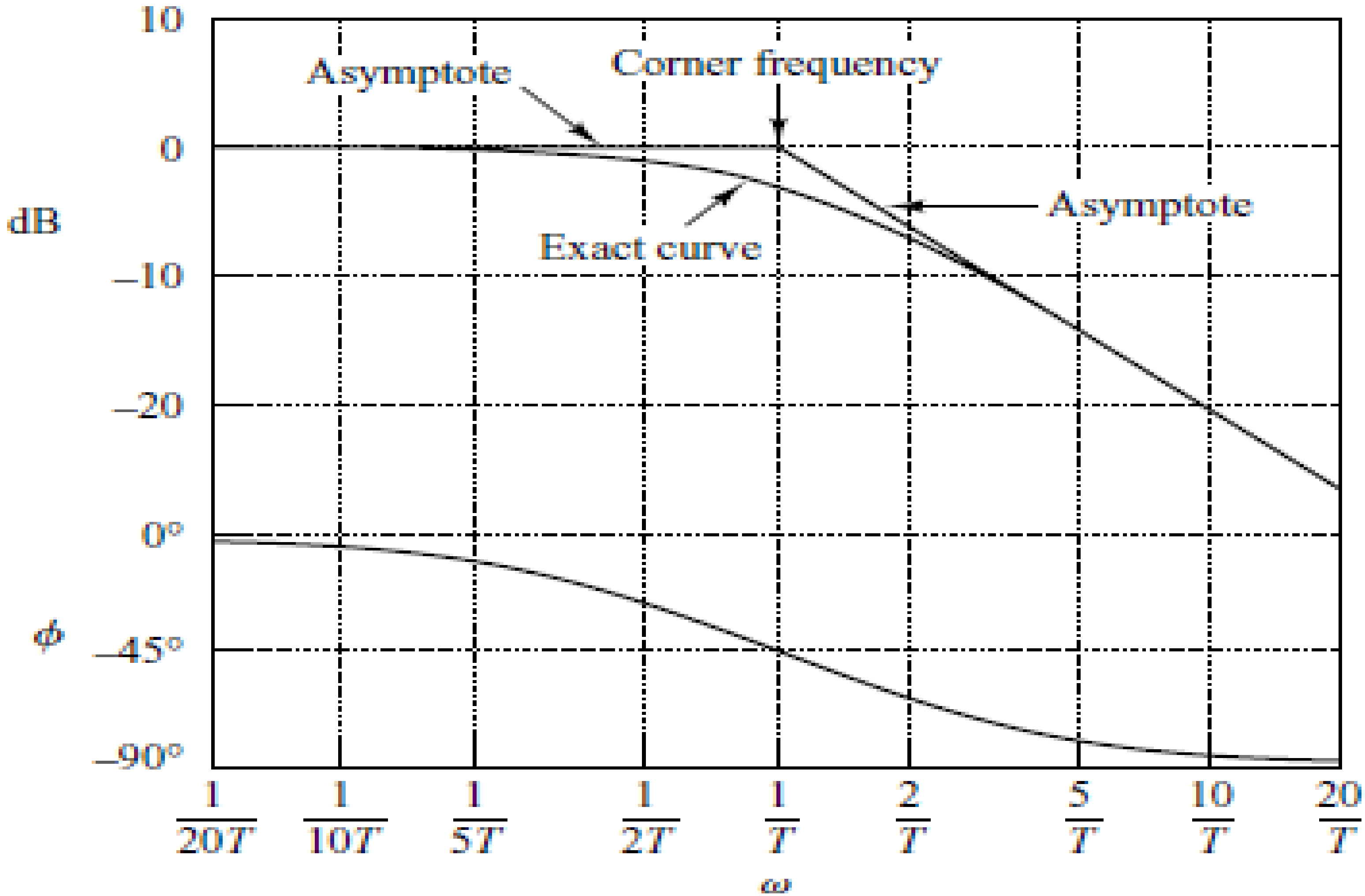
$$= -20\log\sqrt{1+(\omega T)^2}$$

$\omega T << 1$, $M = -20\log 1 = 0$ db $\qquad \emptyset = 0^o$

$\omega T = 1$, $M = -20\log\sqrt{2} = -3$ db $\qquad \emptyset = -45^o$

(corner frequency)

$\omega T >> 1$, $M = -20\log \omega T$ $\qquad \emptyset = -90^o$



10
0
dB
–10
–20
0°
φ
–45°
–90°
Asymptote
Corner frequency
Asymptote
Exact curve
1/20T
1/10T
1/5T
1/2T
1/T
2/T
5/T
10/T
20/T
ω

Zeros not at origin

$G(s) = 1+Ts$

$G(j\omega) = 1+j\omega T$

$M = 20\log|1+j\omega T|$ $\quad\quad\quad$ $\emptyset = \tan^{-1}\omega T$

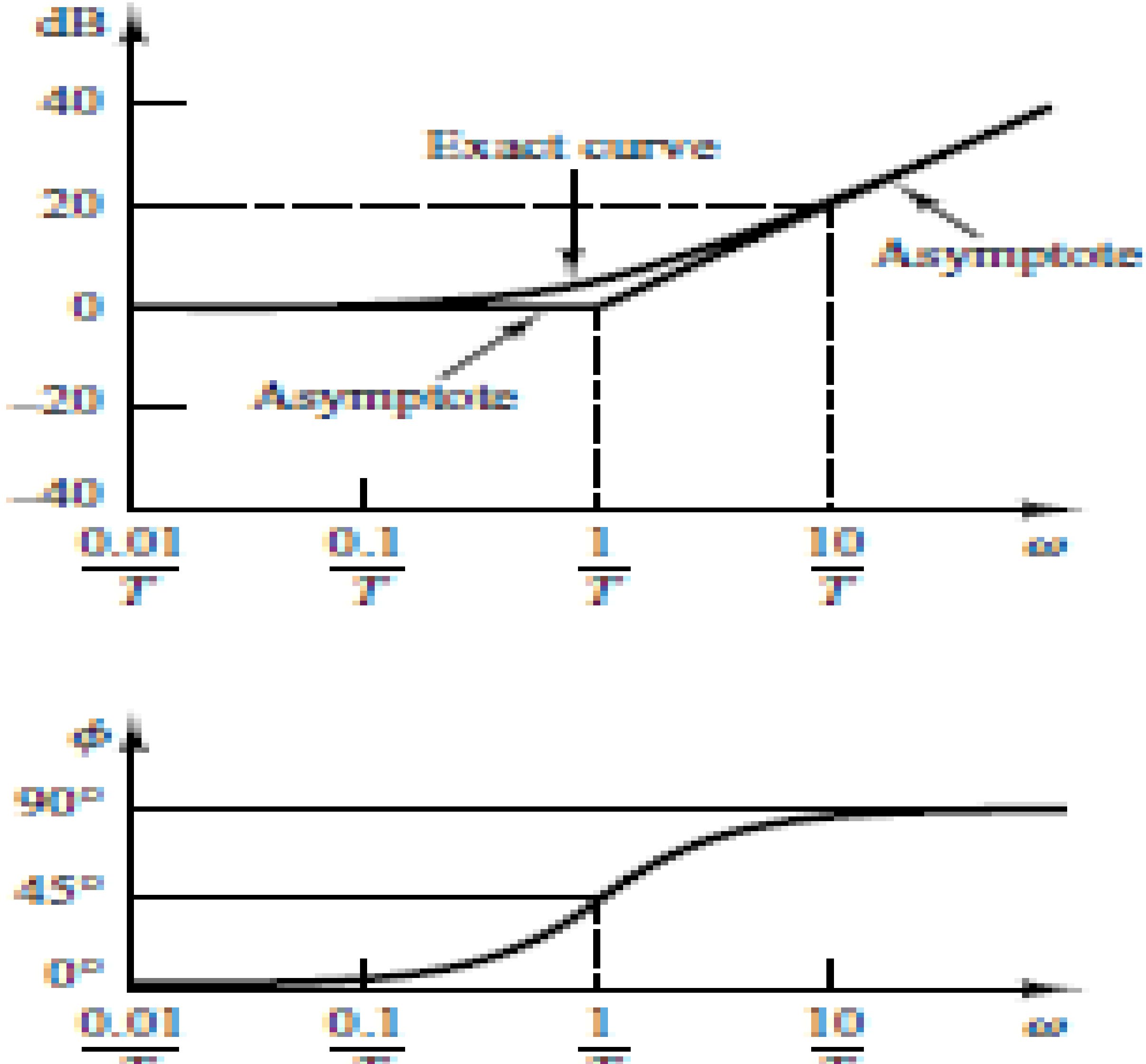
$= 20\log\sqrt{1+(\omega T)^2}$

$\omega T << 1,$ $\quad$ $M = 20\log 1 = 0$ db $\quad\quad$ $\emptyset = 0^o$

$\omega T = 1,$ $\quad$ $M = 20\log\sqrt{2} = 3$ db $\quad\quad$ $\emptyset = 45^o$

(corner frequency)

$\omega T >> 1,$ $\quad$ $M = 20\log\omega T$ $\quad\quad$ $\emptyset = 90^o$


dB
40
20
0
−20
−40
Exact curve
Asymptote
Asymptote
0.01/T
0.1/T
1/T
10/T
ω
ϕ
90°
45°
0°
0.01
0.1
1
10
ω

Complex Poles

$$G(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$S=j\omega$

$$G(j\omega)= \frac{\omega_n^2}{(j\omega)2+2\zeta\omega_n j\omega+\omega_n^2} = \frac{\omega_n^2}{\omega_n^2\left[1+j2\zeta\left(\frac{\omega}{\omega_n}\right)+\left(\frac{j\omega}{\omega_n}\right)^2\right]}$$

$$= \frac{1}{\left[1+j2\zeta\left(\frac{\omega}{\omega_n}\right)-\left(\frac{\omega}{\omega_n}\right)^2\right]} = \frac{1}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2+j2\zeta\left(\frac{\omega}{\omega_n}\right)\right]}$$

$$M = -20\log\left[\sqrt{\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2+\left\{2\zeta\left(\frac{\omega}{\omega_n}\right)\right\}^2}\right]$$

$$\emptyset = -\tan^{-1}\left\{\frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)}\right\}$$

$\underline{\text{Magnitude}}$: $M = -20\log\left[\sqrt{\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2+\left\{2\zeta\left(\frac{\omega}{\omega_n}\right)\right\}^2}\right]$

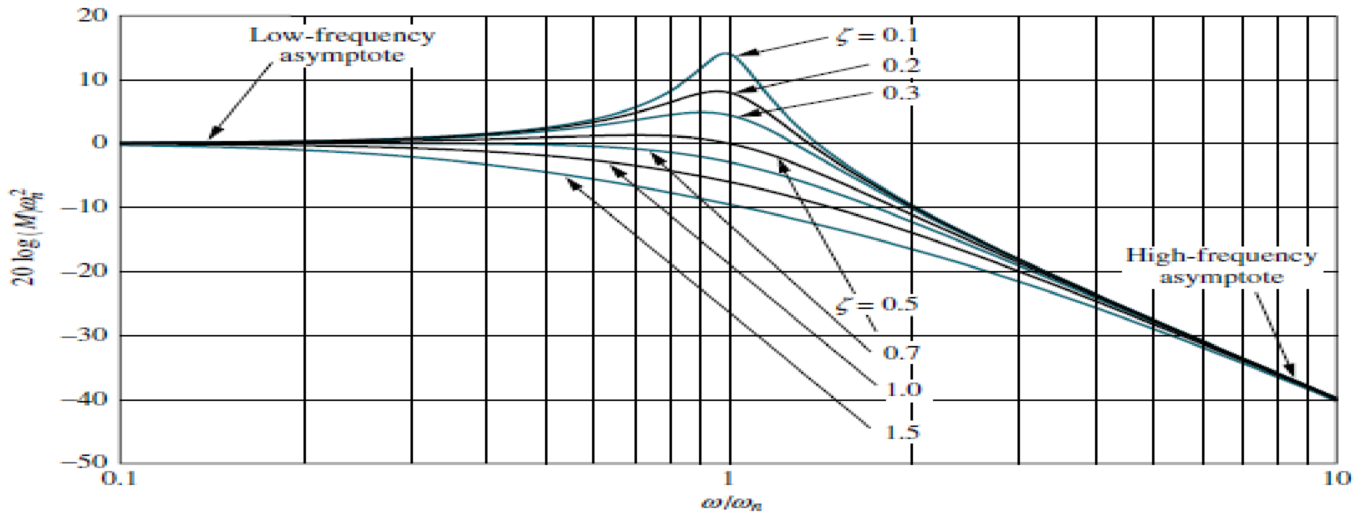
For $\zeta < 1$,

$\omega << \omega_n$, $M = -20\log 1 = 0$ db

$\omega = \omega_n$, $M = -20\log(2\zeta)$

(corner frequency)

$\omega >> \omega_n$, $M = -20\log\left(\frac{\omega}{\omega_n}\right)^2 = -40\log\left(\frac{\omega}{\omega_n}\right)$

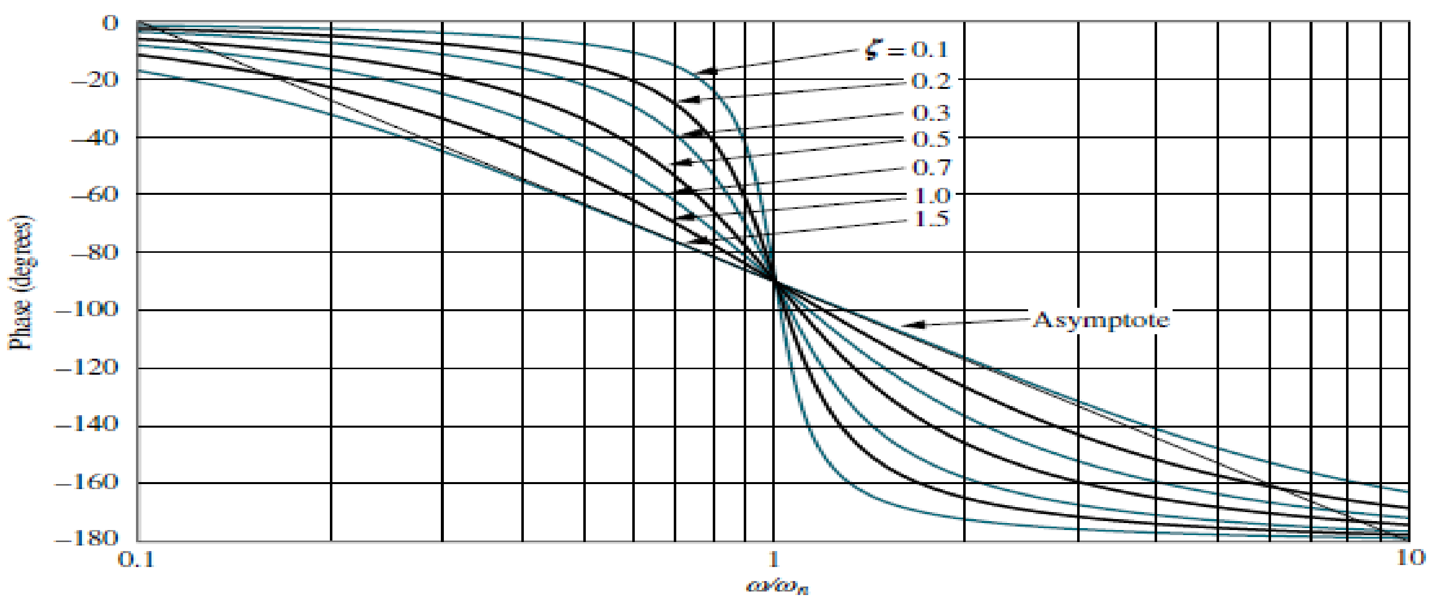


Phase: $\quad \varnothing = -\tan^{-1}\left\{ \frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)} \right\}$

$\omega << \omega_n$, $\quad \varnothing = 0^o$

$\omega = \omega_n$, (corner frequency) $\quad \varnothing = -90^o$

$\omega >> \omega_n$, $\quad \varnothing = -180^o$



Complex Zeros

$G(s) = s^2 + 2\zeta\omega_n s + \omega_n{}^2$

S=jω

$G(j\omega)= [(j\omega)2 + 2\zeta\omega_n j\omega + \omega_n{}^2]$

$$= \omega_n{}^2 \left[1 + j2\zeta\left(\frac{\omega}{\omega_n}\right) + \left(\frac{j\omega}{\omega_n}\right)^2\right]$$

$$= \omega_n{}^2 \left[1 + j2\zeta\left(\frac{\omega}{\omega_n}\right) - \left(\frac{\omega}{\omega_n}\right)^2\right]$$

$$M = 20\log \surd \left[\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left\{2\zeta\left(\frac{\omega}{\omega_n}\right)\right\}^2\right]$$

$$\varnothing = \tan^{-1}\left\{ \frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)}\right\}$$

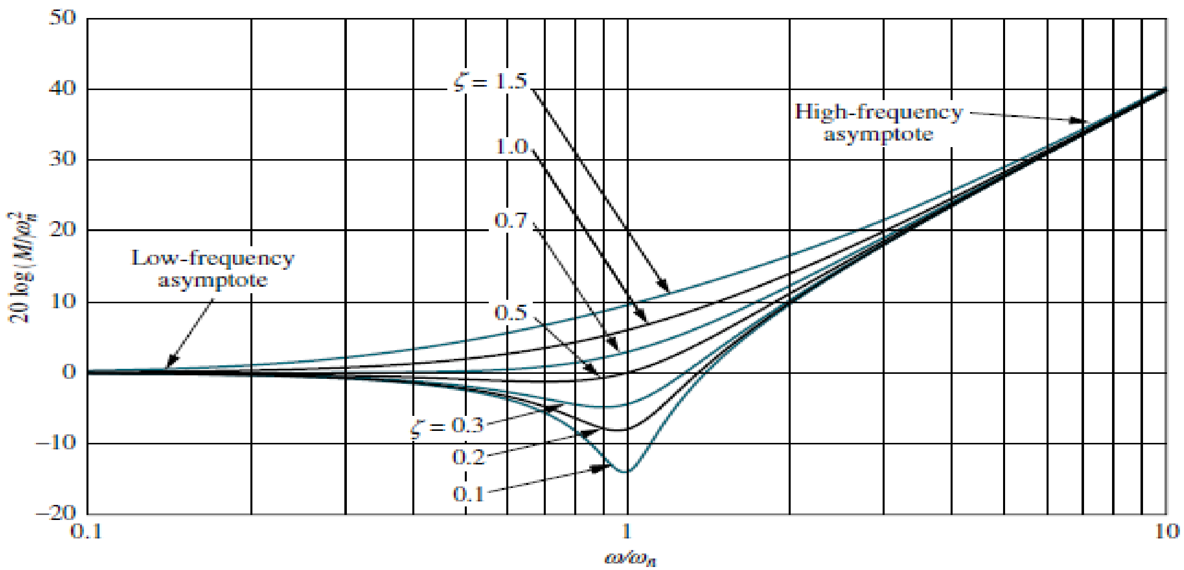
$$\underline{\text{Magnitude}}: M = 20\log\left[\sqrt{\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2+\left\{2\zeta\left(\frac{\omega}{\omega_n}\right)\right\}^2}\right]$$

For $\zeta < 1$,

$\omega << \omega_n$, $\quad M = 20\log 1 = 0$ db

$\omega = \omega_n$, $\quad M = 20\log(2\zeta)$ $\quad$ (corner frequency)

$\omega >> \omega_n$, $\quad M = 20\log\left(\frac{\omega}{\omega_n}\right)^2 = 40\log\left(\frac{\omega}{\omega_n}\right)$

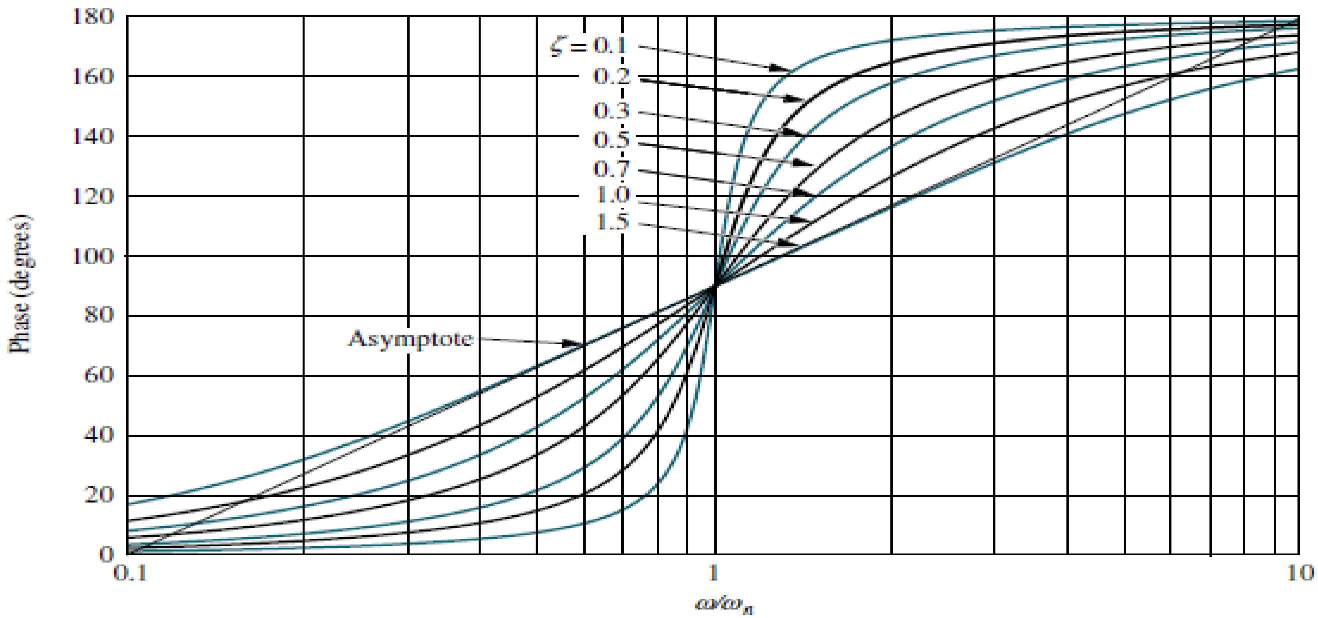


Phase: $\varnothing = \tan^{-1}\left\{\frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right.}\right\}$

$\omega << \omega_n, \ \varnothing = 0^o$

$\omega = \omega_n, \quad \varnothing = 90^o$ (corner frequency)

$\omega >> \omega_n, \ \varnothing = 180^o$



Problem 1

Draw the Bode plot for the transfer function $G(s) = \frac{10}{s(1+0.5s)(1+0.1s)}$

Solution:

$$G(s) = \frac{10}{s(1+0.5s)(1+0.1s)}$$

$$s = j\omega$$

$$G(j\omega) = \frac{10}{j\omega(1+0.5j\omega)(1+0.1j\omega)}$$

$$G(j\omega) = |G(j\omega)|\angle G(j\omega)$$

$$|G(j\omega)| = \left|\frac{10}{j\omega(1+0.5j\omega)(1+0.1j\omega)}\right|$$

$$|\angle G(j\omega) = \angle\frac{10}{j\omega(1+0.5j\omega)(1+0.1j\omega)}$$

$$|G(j\omega)| = \left|\frac{10}{j\omega(1+0.5j\omega)(1+0.1j\omega)}\right|$$

Magnitude (db) = 20log $|G(j\omega)|$

$$M_{db} = 20\log 10 - 20\log \omega - 20\log \sqrt{(1+(0.5\omega)^2)} -$$

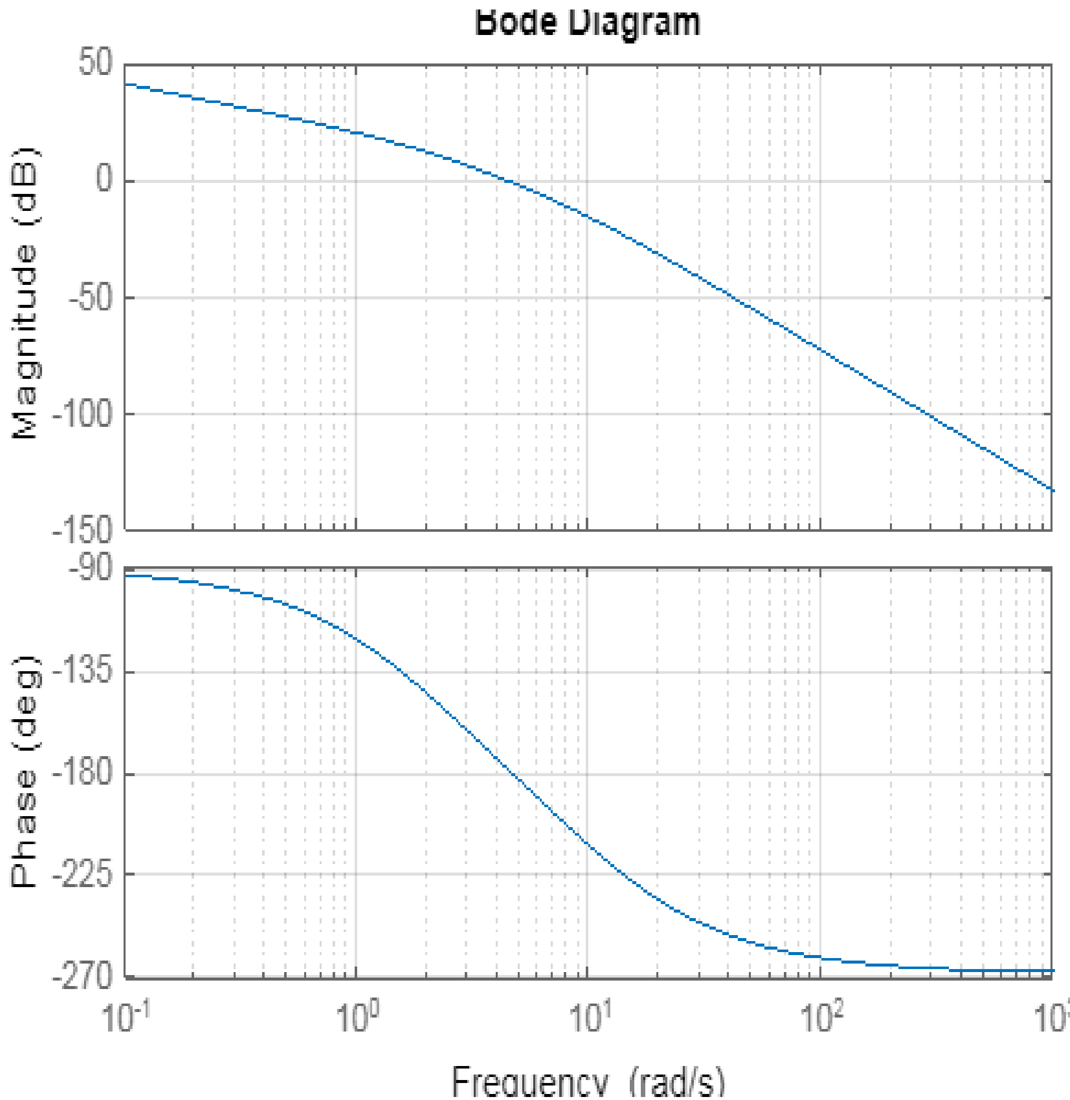
$$20\log \sqrt{(1+(0.1\omega)^2)}$$

$$\angle G(j\omega) = \angle\frac{10}{j\omega(1+0.5j\omega)(1+0.1j\omega)}$$

$$\emptyset = \tan^{-1}\left(\frac{0}{10}\right) - \tan^{-1}\left(\frac{\omega}{0}\right) - \tan^{-1}\left(\frac{0.5\omega}{1}\right) - \tan^{-1}\left(\frac{0.1\omega}{1}\right)$$

$$= 0 - 90 - \tan^{-1}(0.5\omega) - \tan^{-1}(0.1\omega)$$

ω	M (db)	Ø (deg)
0.1	40	-93
1	19	-122.3
2	11	-146
10	-17.16	-213.7
100	-74	-263



Problem 2

Draw the Bode plot for the transfer function $G(s)= \frac{(s+3)}{s(s+1)(s+2)}$

$$G(s)= \frac{(s+3)}{s(s+1)(s+2)} = \frac{3(\frac{s}{3}+1)}{s(s+1)2(\frac{s}{2}+1)} = \frac{1.5(1+0.3s)}{s(s+1)(1+0.5s)}$$

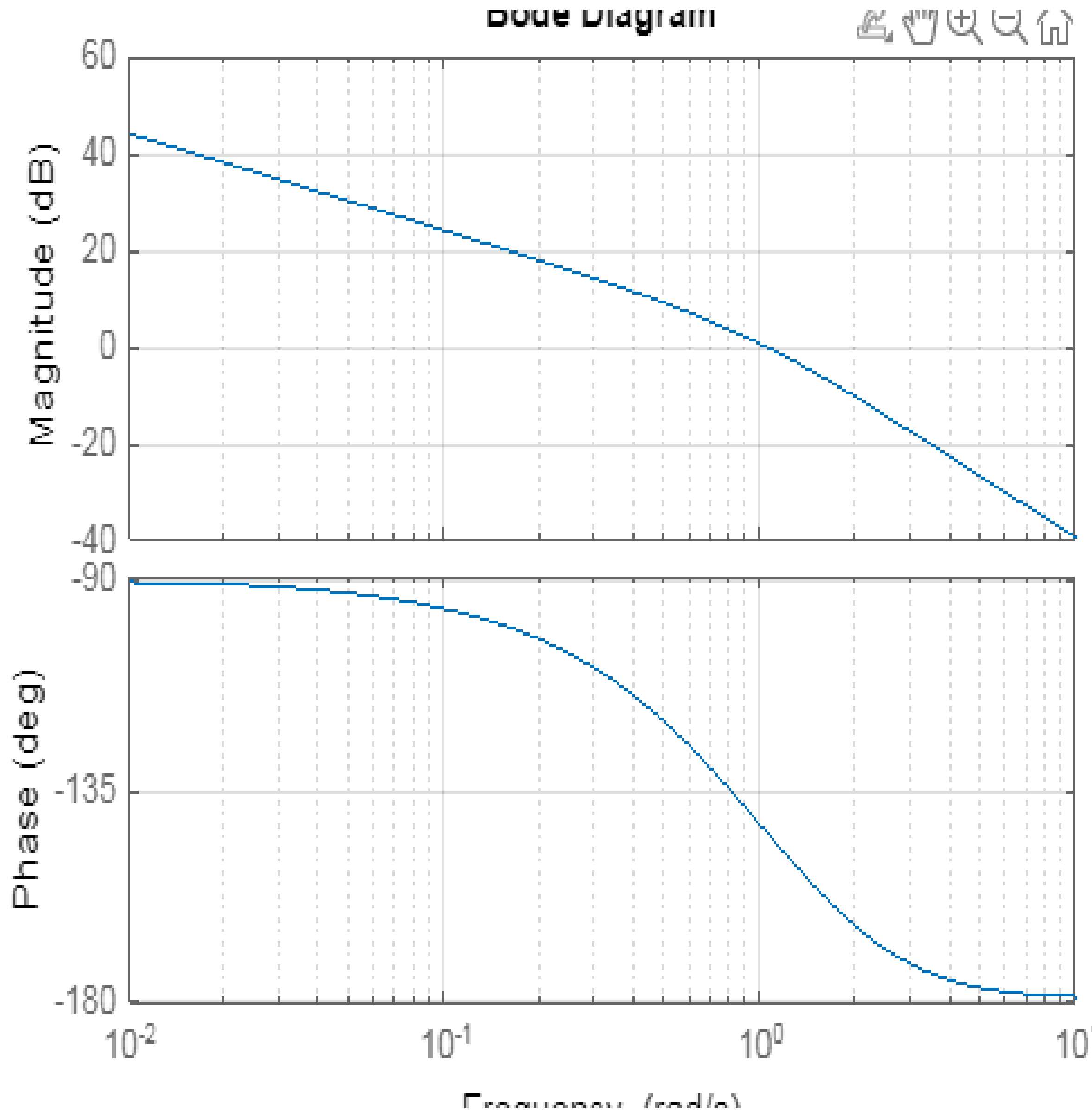
$$G(j\omega) = \frac{1.5(1+0.3j\omega)}{j\omega(j\omega+1)(1+0.5j\omega)}$$

$$M_{db} = 20\log 1.5 + 20\log \sqrt{(1+(0.3\omega)^2} - 20\log \omega - 20\log \sqrt{(1+(\omega)^2} - 20\log \sqrt{(1+(0.5\omega)^2}$$

$$\emptyset = \tan^{-1}(\frac{0}{1.5}) + \tan^{-1}(\frac{0.3\omega}{1}) - \tan^{-1}(\frac{\omega}{0}) - \tan^{-1}(\frac{\omega}{1}) - \tan^{-1}(\frac{0.5\omega}{1})$$

$$\emptyset = 0 + \tan^{-1}(0.3\omega) - 90 - \tan^{-1}\omega - \tan^{-1}(0.5\omega)$$

ω	M	Ø
0.1	23.5	-97
1	-0.08	-145
2	-11.16	-167
3	-18.6	-176
10	-40.7	-181
100	-81	-180



Problem 3

The open loop transfer function of a unity feedback system is given by $\frac{64(s+2)}{s(s+0.5)(s^2+3.2s+64)}$. Draw the Bode plot and hence comment on stability.

$$G(s)=\frac{64(s+2)}{s(s+0.5)(s^2+3.2s+64)} = \frac{64*2\ (\frac{s}{2}+1)}{s*0.5\left(\frac{s}{0.5}+1\right)*64*(\frac{s^2}{64}+\frac{3.2s}{64}+1)}$$

$$=\frac{4(0.5s+1)}{s(2s+1)(0.015s^2+0.05s+1)}$$

$$S=j\omega$$

$$G(j\omega) = \frac{4(0.5j\omega+1)}{j\omega(2j\omega+1)(0.015(j\omega)^2+0.05j\omega+1)} = \frac{4(0.5j\omega+1)}{j\omega(2j\omega+1)(1-0.015\omega^2+0.05j\omega)}$$

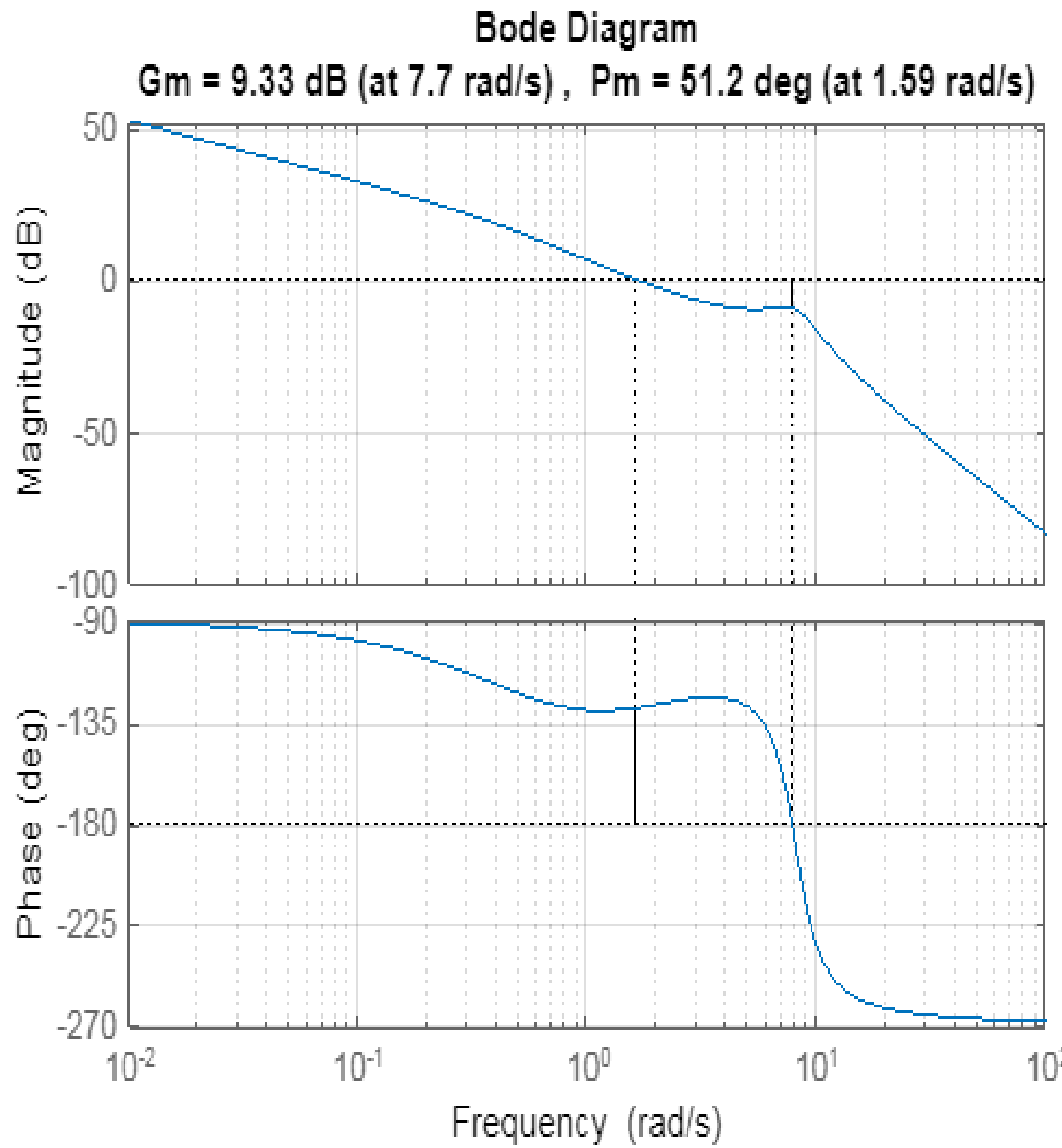
$$M_{db} = 20\log 4 + 20\log \sqrt{(1+(0.5\omega)^2\ )} - 20\log \omega -$$

$$20\log \sqrt{(1+(2\omega)^2\ )} - 20\log \sqrt{\{(1-0.015\omega^2)^2+(0.05\omega)^2\ \}}$$

$$\varnothing = \tan^{-1}(\frac{0}{4})+\tan^{-1}(\frac{0.5\omega}{1}) -\tan^{-1}(\frac{\omega}{0}) -\tan^{-1}(\frac{2\omega}{1}) -\tan^{-1}(\frac{0.05\omega}{1-0.015\omega^2})$$

$$=0 + \tan^{-1} 0.5\omega - 90 - \tan^{-1} 2\omega - \tan^{-1}(\frac{0.05\omega}{1-0.015\omega^2})$$

ω	M	Ø
0.1	32	-99
0.5	15	-122
1	6	-130
2	-3	-127
3	-7	-124
8	-9	-185
10	-17	-53(-180) = -233
100	-83	-89 (-180) =-269



Problem 4

Draw the Bode plot for the transfer function $G(s)=\frac{10(s+3)}{s(s+2)(s^2+2s+2)}$

$$G(s)=\frac{10(s+3)}{s(s+2)(s^2+2s+2)}=\frac{10*3\ (\frac{s}{3}+1)}{s*2\left(\frac{s}{2}+1\right)*2*(\frac{s^2}{2}+\frac{2s}{2}+1)}=\frac{7.5(0.33s+1)}{s(0.5s+1)(0.5s^2+s+1)}$$

$$S=j\omega$$

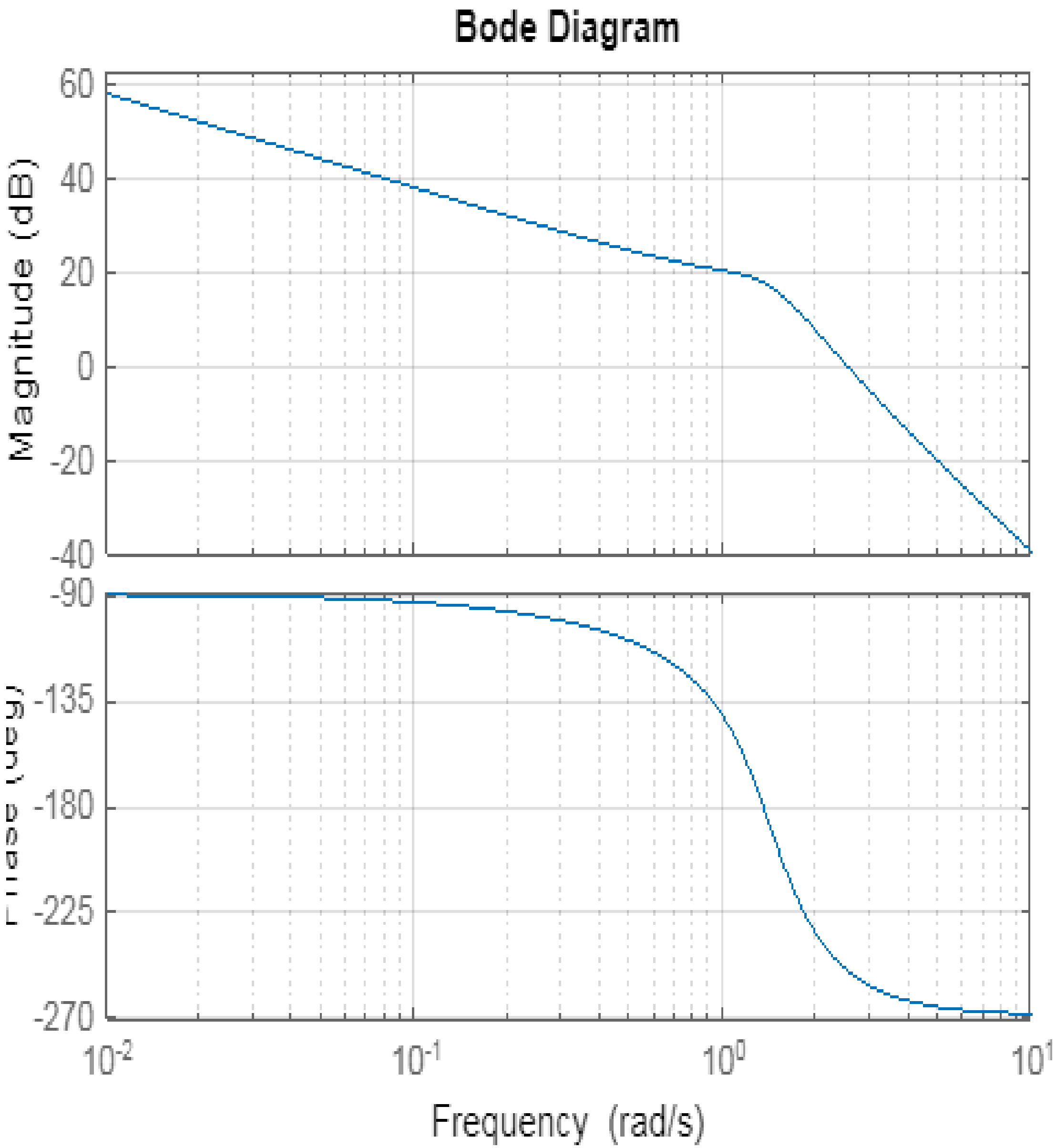
$$G(j\omega)=\frac{7.5(0.33j\omega+1)}{j\omega(0.5j\omega+1)(0.5(j\omega)^2+j\omega+1)}=\frac{7.5(0.33j\omega+1)}{j\omega(0.5j\omega+1)(1-0.5\omega^2+j\omega)}$$

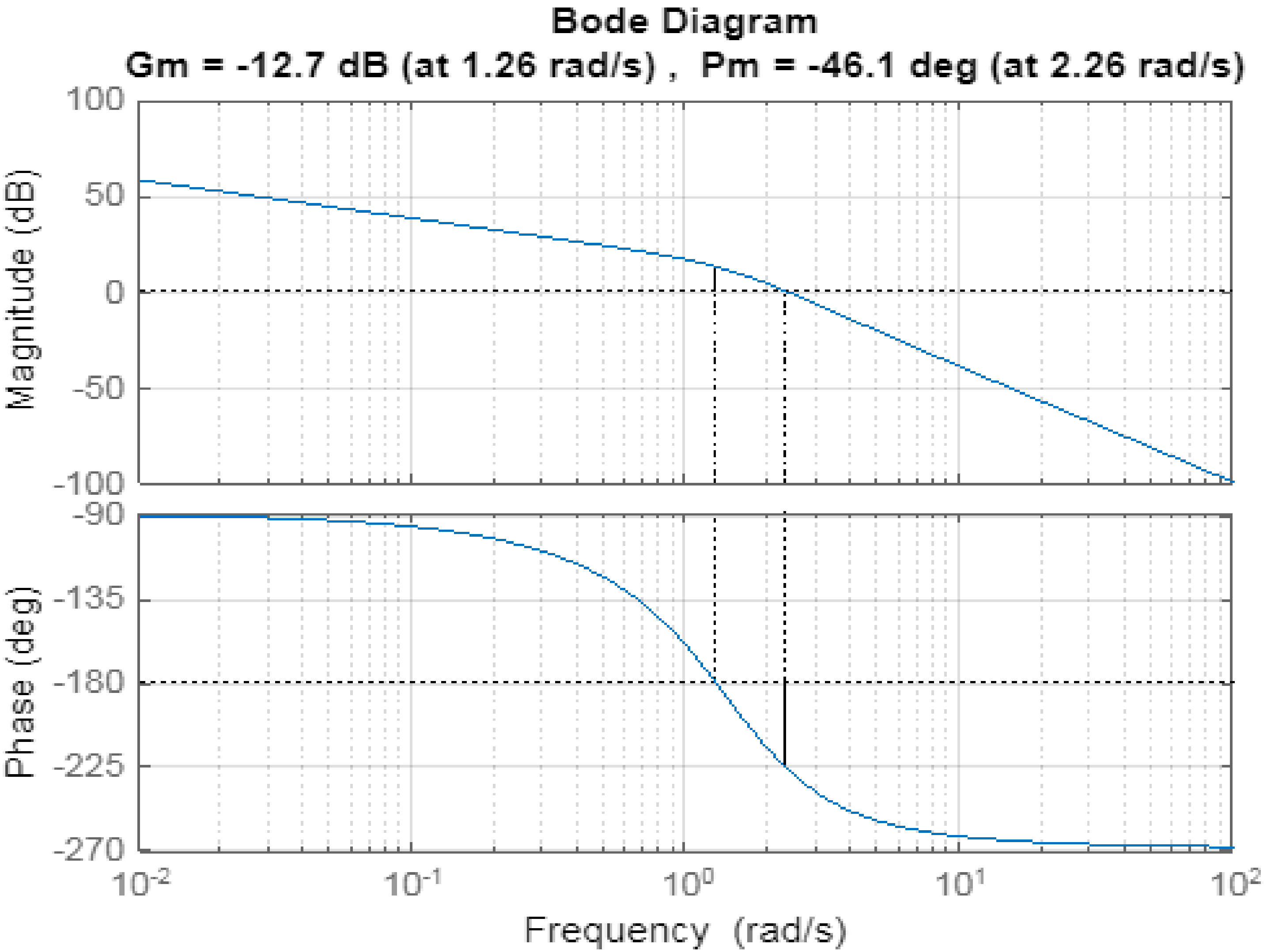
$$M_{db}=20\log 7.5+20\log\sqrt{(1+(0.33\omega)^2}\ -20\log\omega-$$
$$20\log\sqrt{(1+(0.5\omega)^2}-20\log\sqrt{\{(1-0.5\omega^2)^2+\omega^2\}}$$

$$\emptyset=\tan^{-1}(\frac{0}{7.5})+\tan^{-1}(\frac{0.33\omega}{1})\ -\tan^{-1}(\frac{\omega}{0})\ -\tan^{-1}(\frac{0.5\omega}{1})\ -\tan^{-1}(\frac{\omega}{1-0.5\omega^2})$$

$$=0+\tan^{-1}0.33\omega-90-\tan^{-1}0.5\omega-\tan^{-1}(\frac{\omega}{1-0.5\omega^2})$$

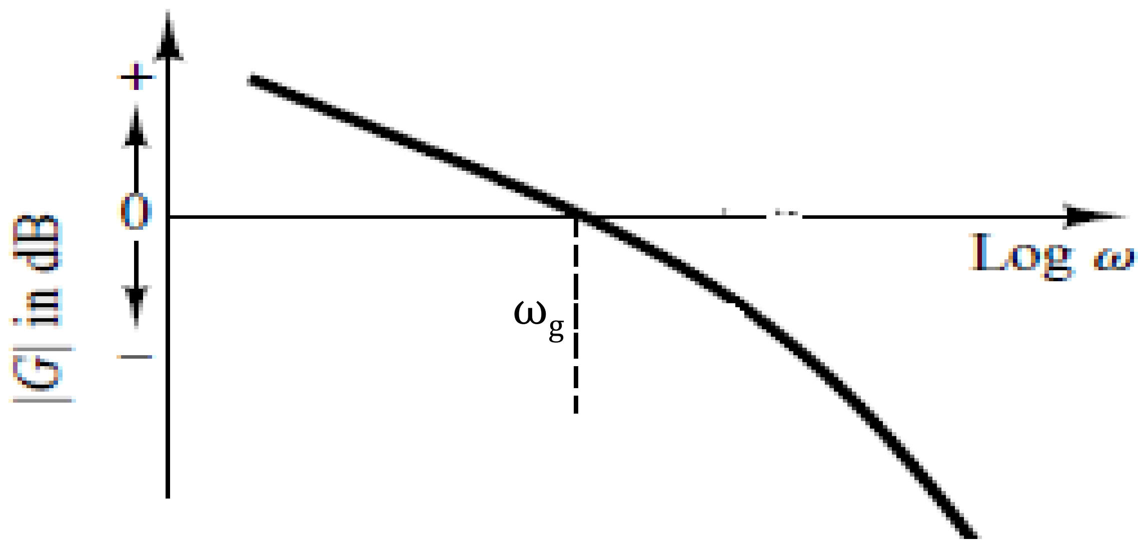
ω	M	Ø
0.1	63.5	-96.7
1	23	-161.7
2	6	-38(- 180) -218
3	-5.9	-61(-180) -241
10	-39.7	-84(-180) -264
100	-100	-89.4(-180) -269.4




Bode Diagram
Gm = -12.7 dB (at 1.26 rad/s) , Pm = -46.1 deg (at 2.26 rad/s)
Magnitude (dB)
100
50
0
-50
-100
Phase (deg)
-90
-135
-180
-225
-270
10^-2
10^-1
10^0
10^1
10^2
Frequency (rad/s)

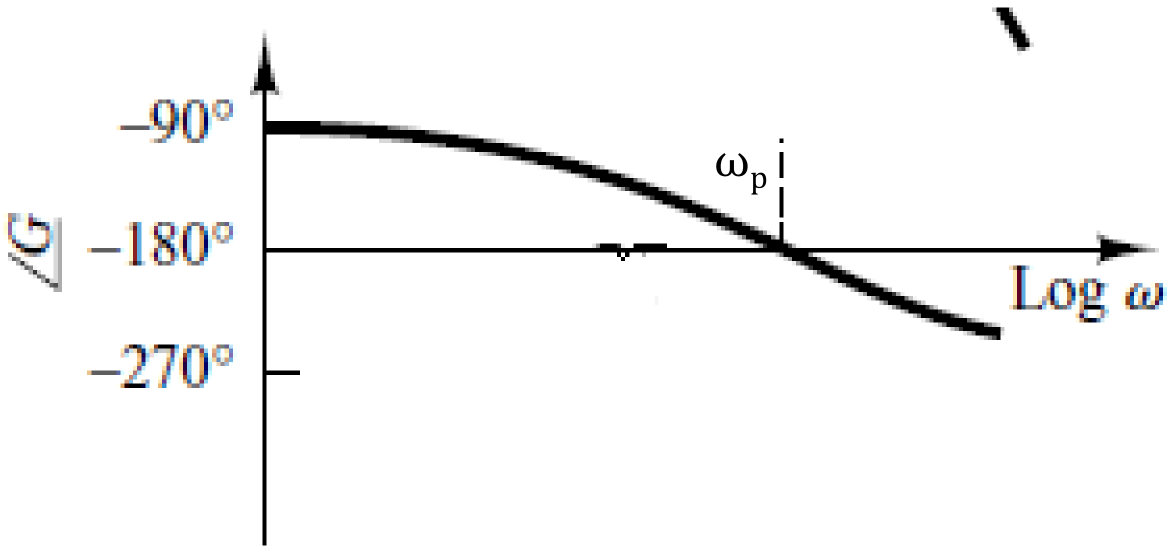
Gain Crossover frequency

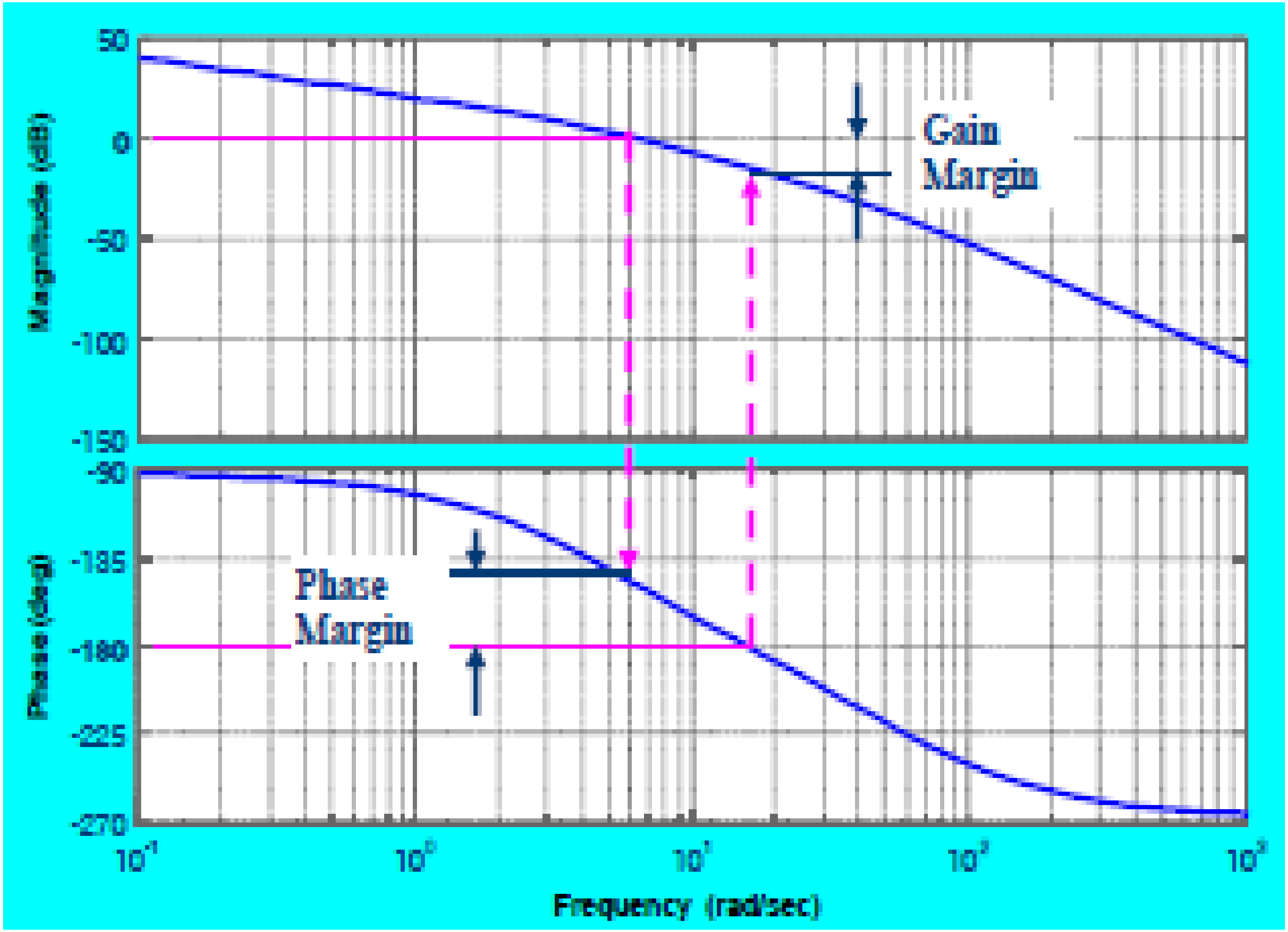
The gain crossover frequency is the frequency at which the magnitude of the open loop transfer function, $|G(j\omega)|$ is unity. In logarithmic scale this corresponds to zero db



Phase Cross over frequency

The phase crossover frequency is the frequency at which the phase of the open loop transfer function, $\angle G(j\omega)$ is -180°.





50
0
-50
-100
-150
Magnitude (dB)
Gain
Margin
-90
-135
-180
-225
-270
Phase (deg)
Phase
Margin
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/sec)

Gain margin

- The gain margin is the amount of additional gain required at the phase crossover frequency to bring the system to the verge of instability.
- The gain margin is the reciprocal of the magnitude $|G(j\omega)|$ at the phase crossover frequency.

$$GM = \frac{1}{|G(j\omega_p)|}$$

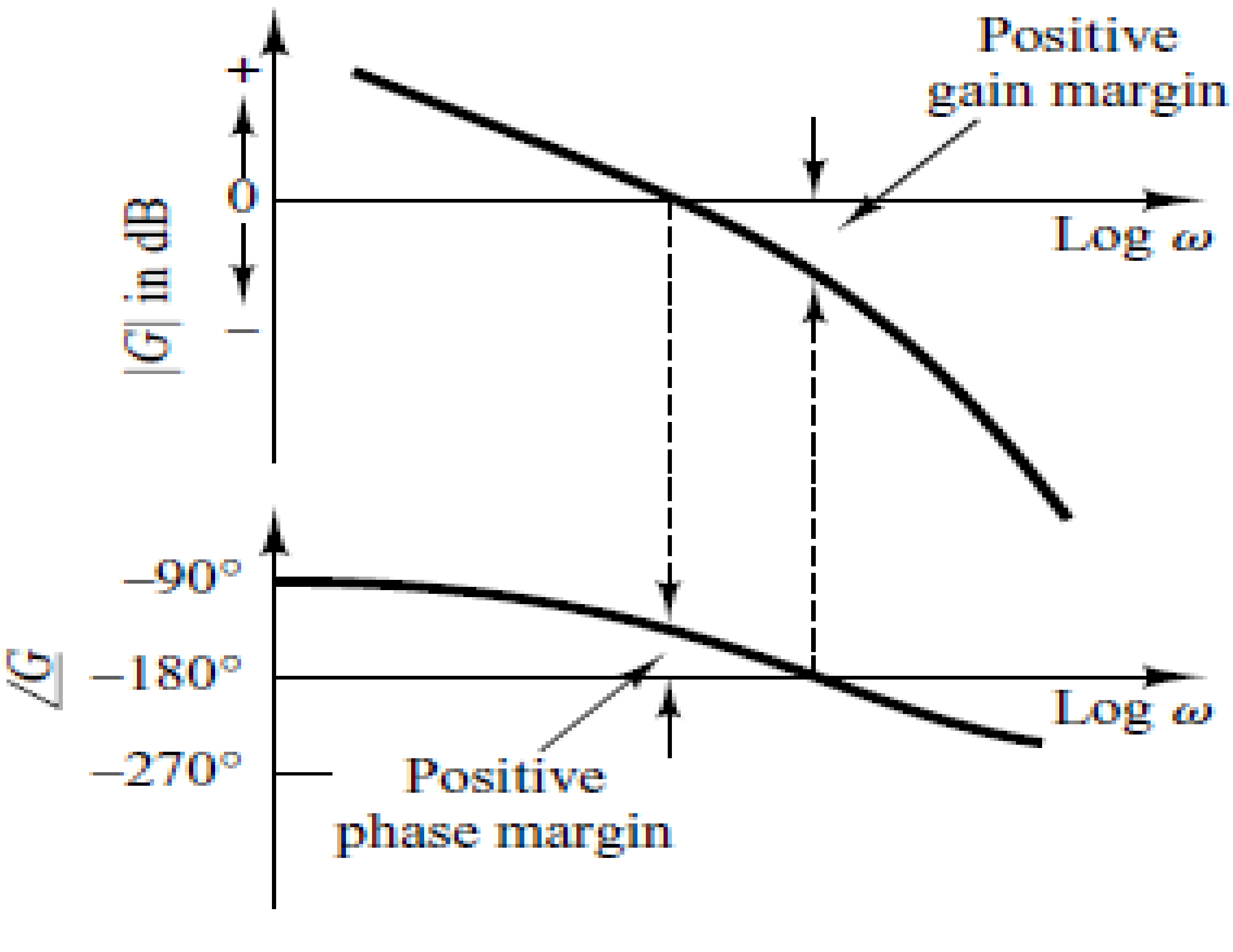
In decibels, $GM = -20\log|G(j\omega_p)|$

Phase margin

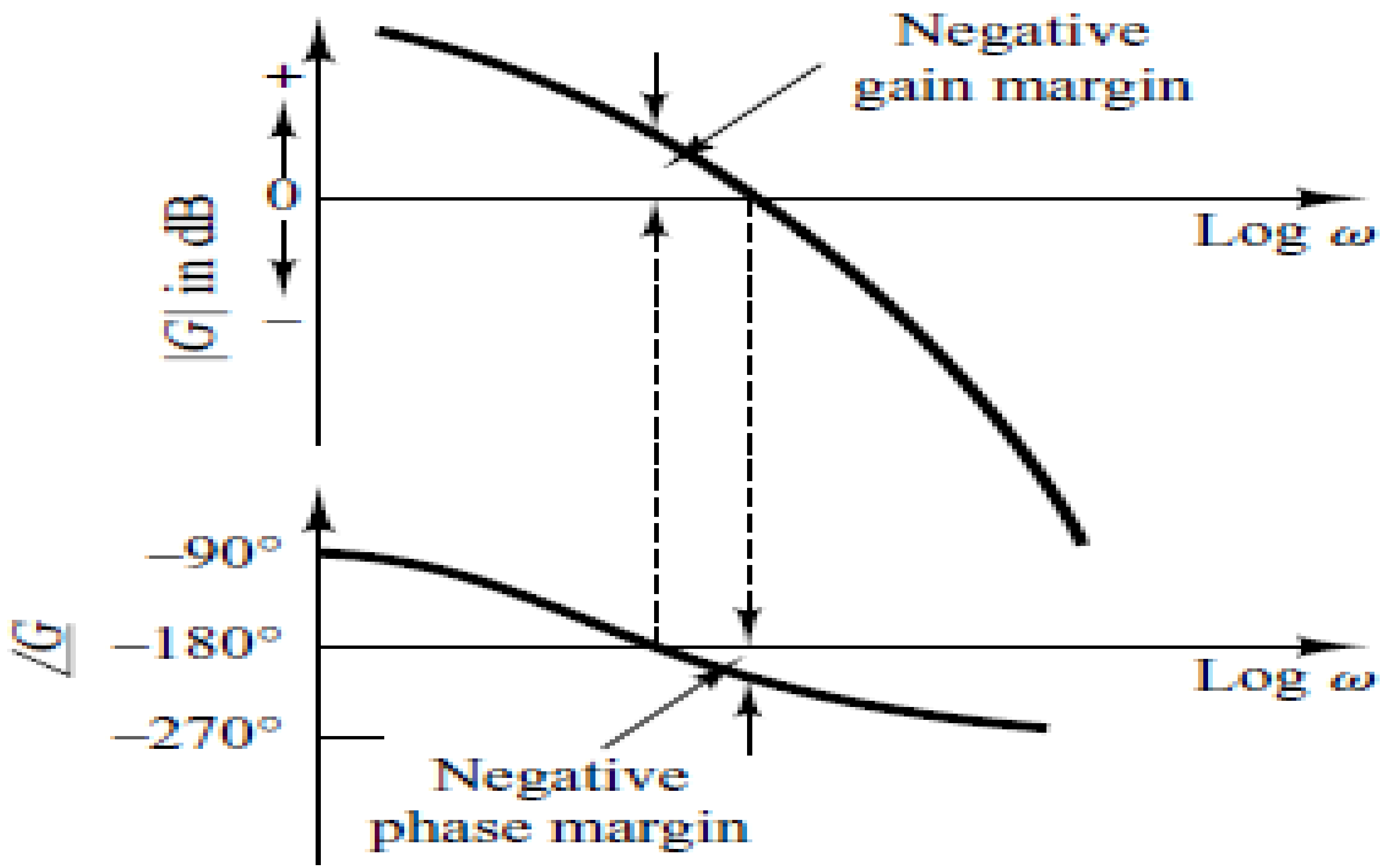
- The phase margin is the amount of additional phase lag required at the gain crossover frequency to bring the system to the verge of instability.
- The phase margin is 180° plus the phase angle of the open-loop transfer function at the gain crossover frequency ω_g

$$PM = 180° + \phi$$

Stable System



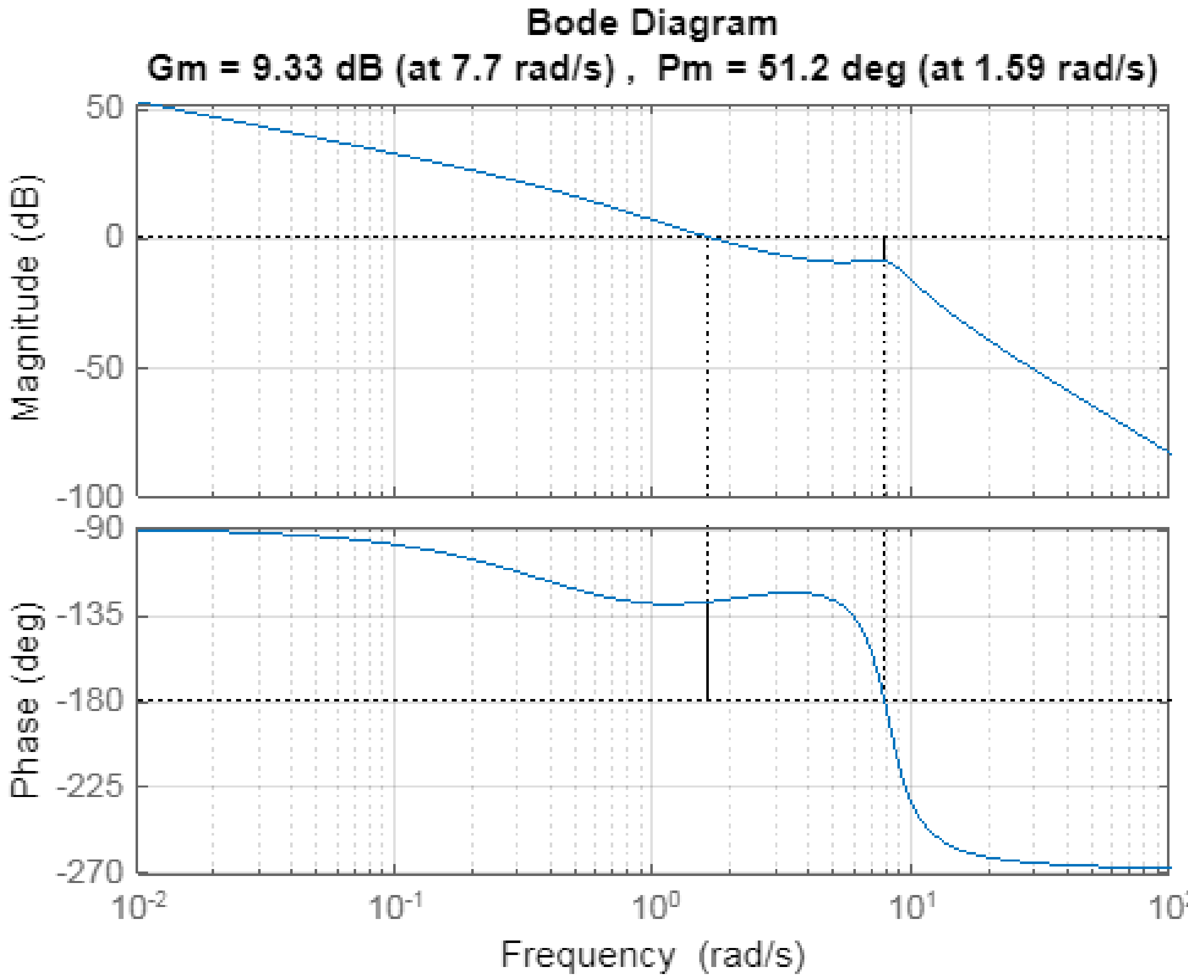
Unstable System



Problem

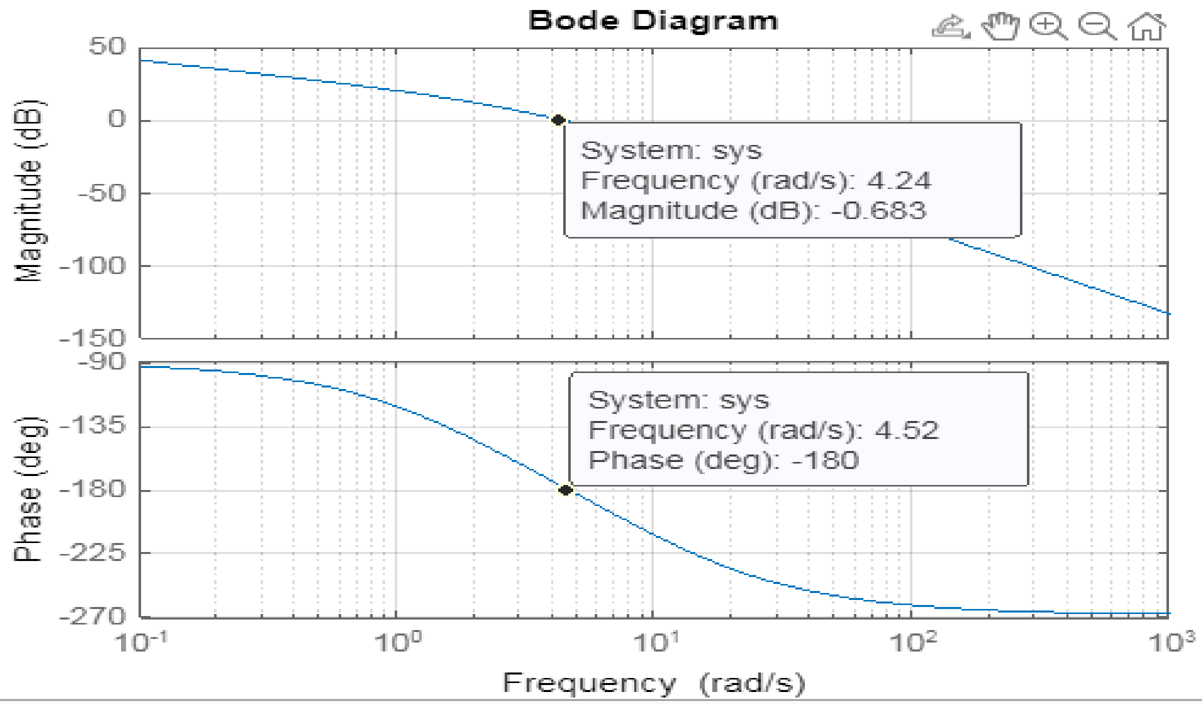
The open loop transfer function of a unity feedback system is given by $\frac{64(s+2)}{s(s+0.5)(s^2+3.2s+64)}$. Draw the Bode plot and hence comment on stability.

```
num=[64 128];
den=[1 3.7 65.6 32 0] ;
bode(num,den)
margin(num,den)
```

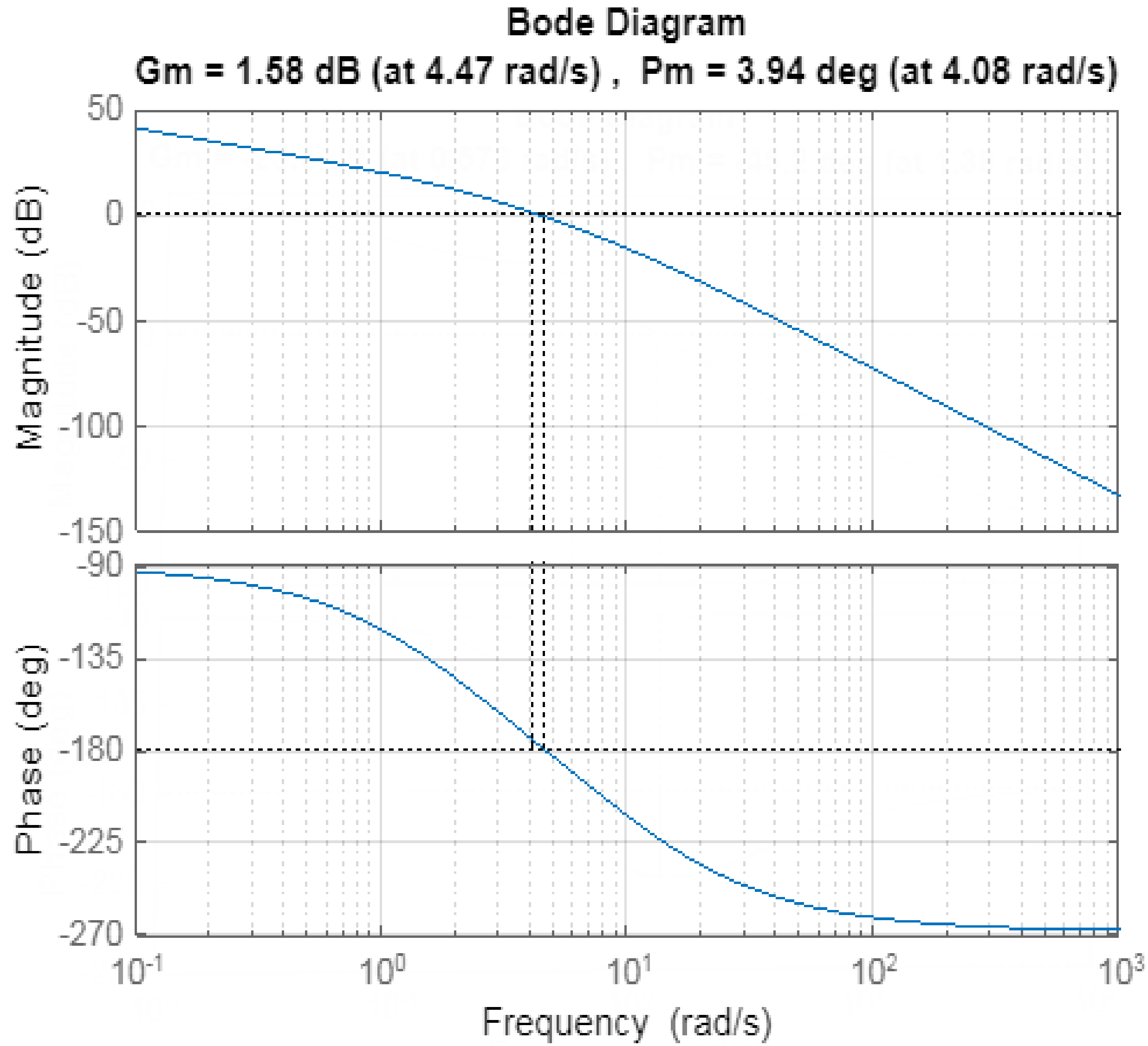


Problem

Draw the Bode plot for the transfer function $G(s) = \dfrac{10}{s(1+0.5s)(1+0.1s)}$ and hence comment on stability.

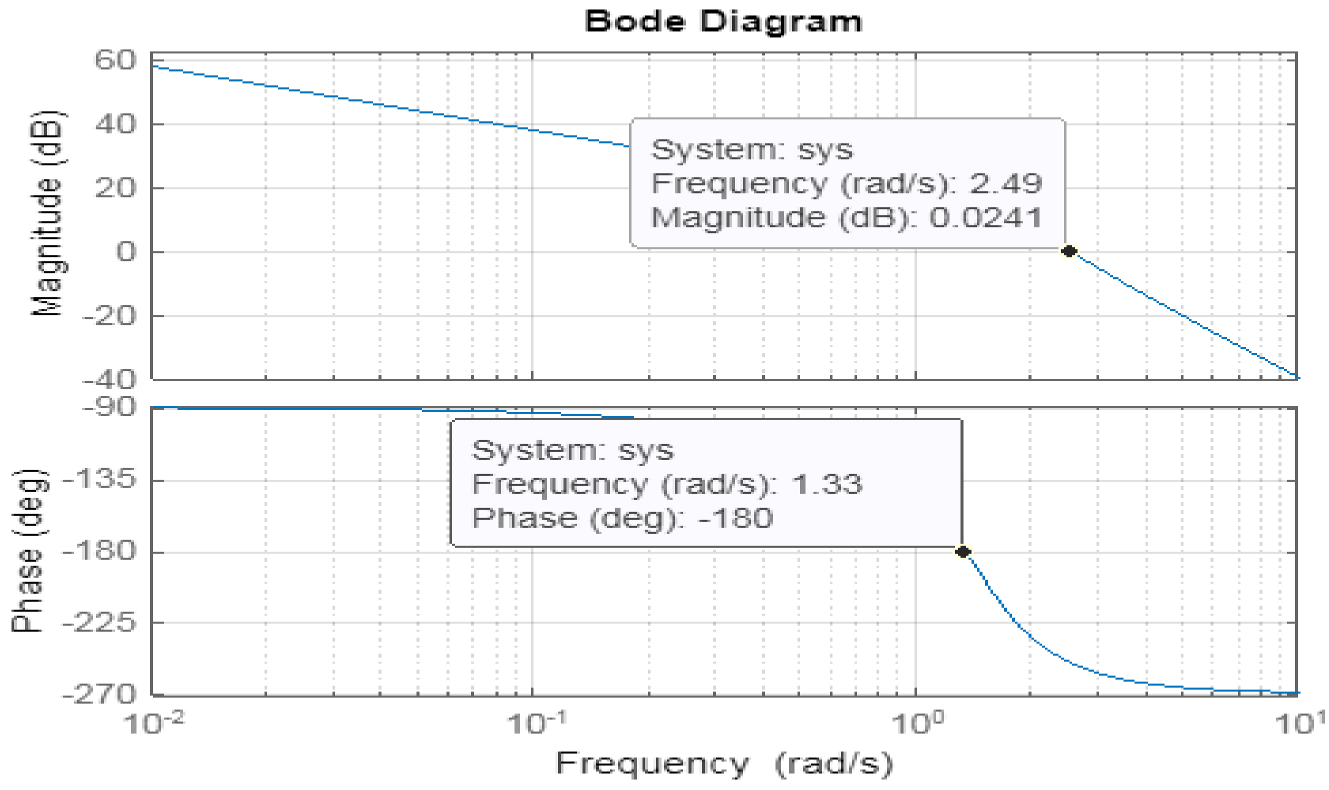


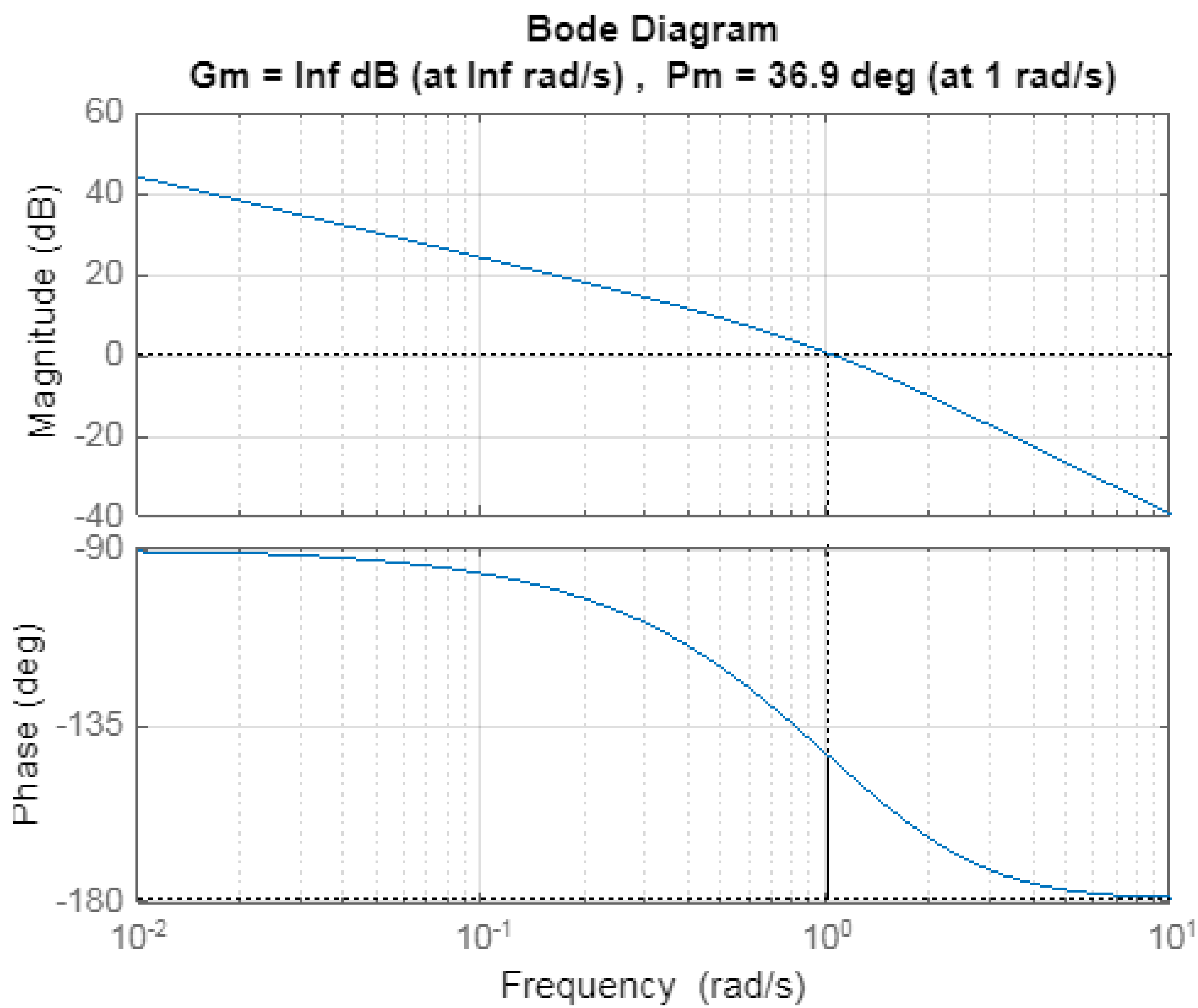
```
num=[10];
den=[0.05 0.6 1 0];
bode(num,den)
margin(num,den)
grid on
```



Problem

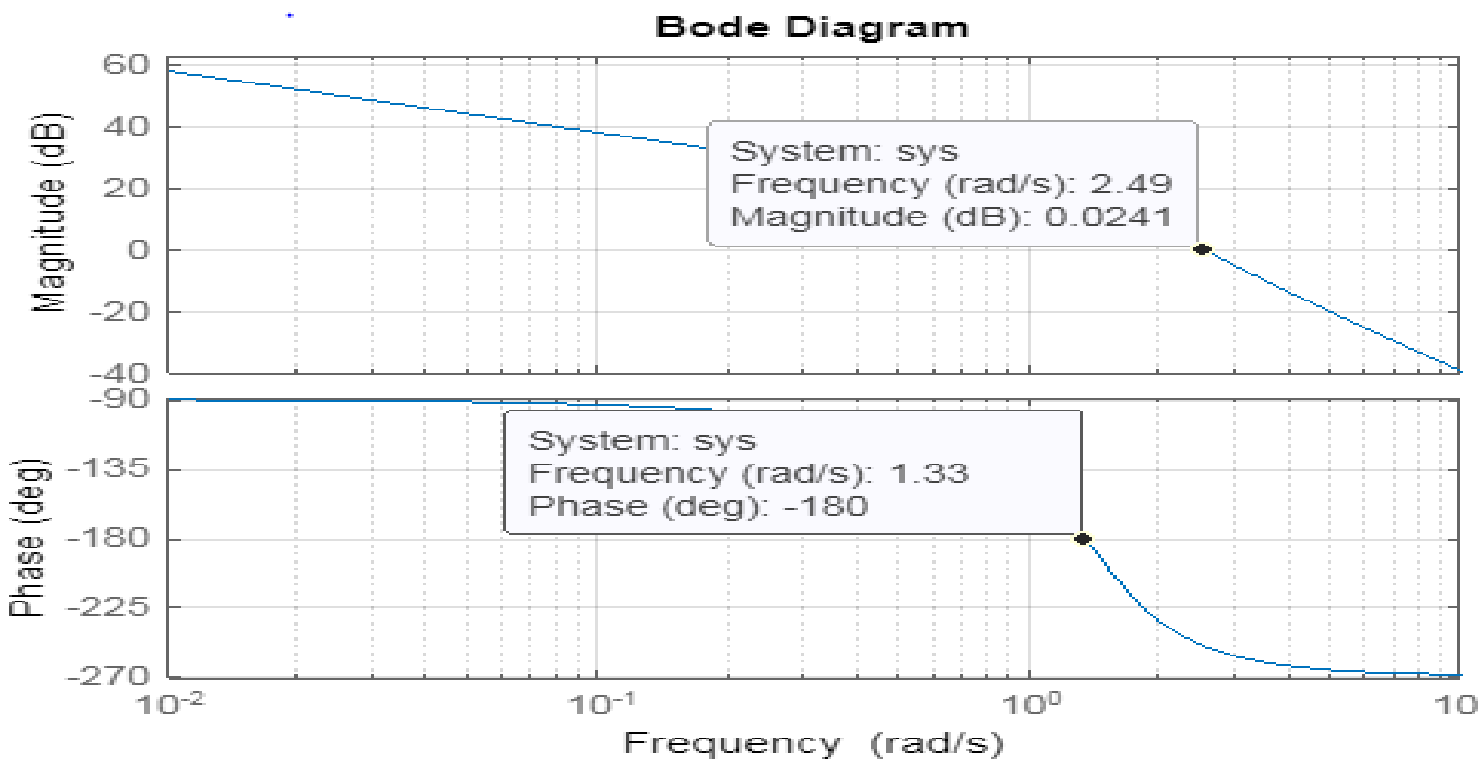
Draw the Bode plot for the transfer function G(s)= $\dfrac{(s+3)}{s(s+1)(s+2)}$ and hence comment on stability.

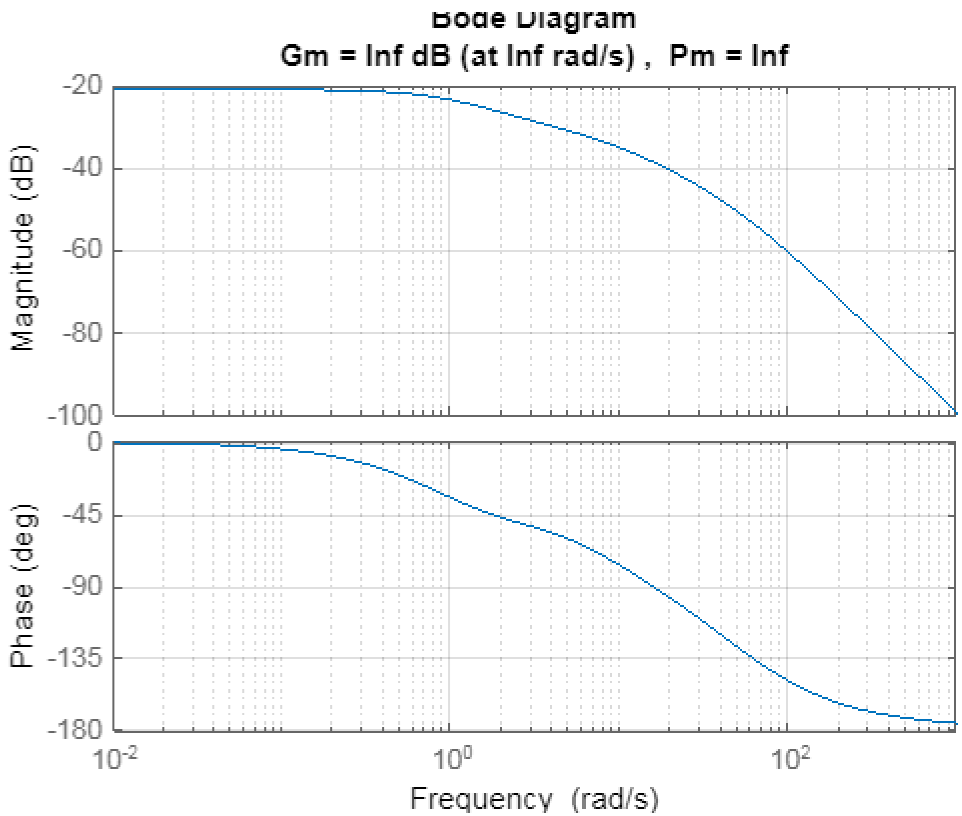



Bode Diagram
Gm = Inf dB (at Inf rad/s) , Pm = 36.9 deg (at 1 rad/s)
Magnitude (dB)
60
40
20
0
-20
-40
Phase (deg)
-90
-135
-180
10-2
10-1
100
101
Frequency (rad/s)

Problem 5

Comment on stability for the following transfer function using Bode plot $\dfrac{10(s+3)}{(s+1)(s+7)(s+50)}$

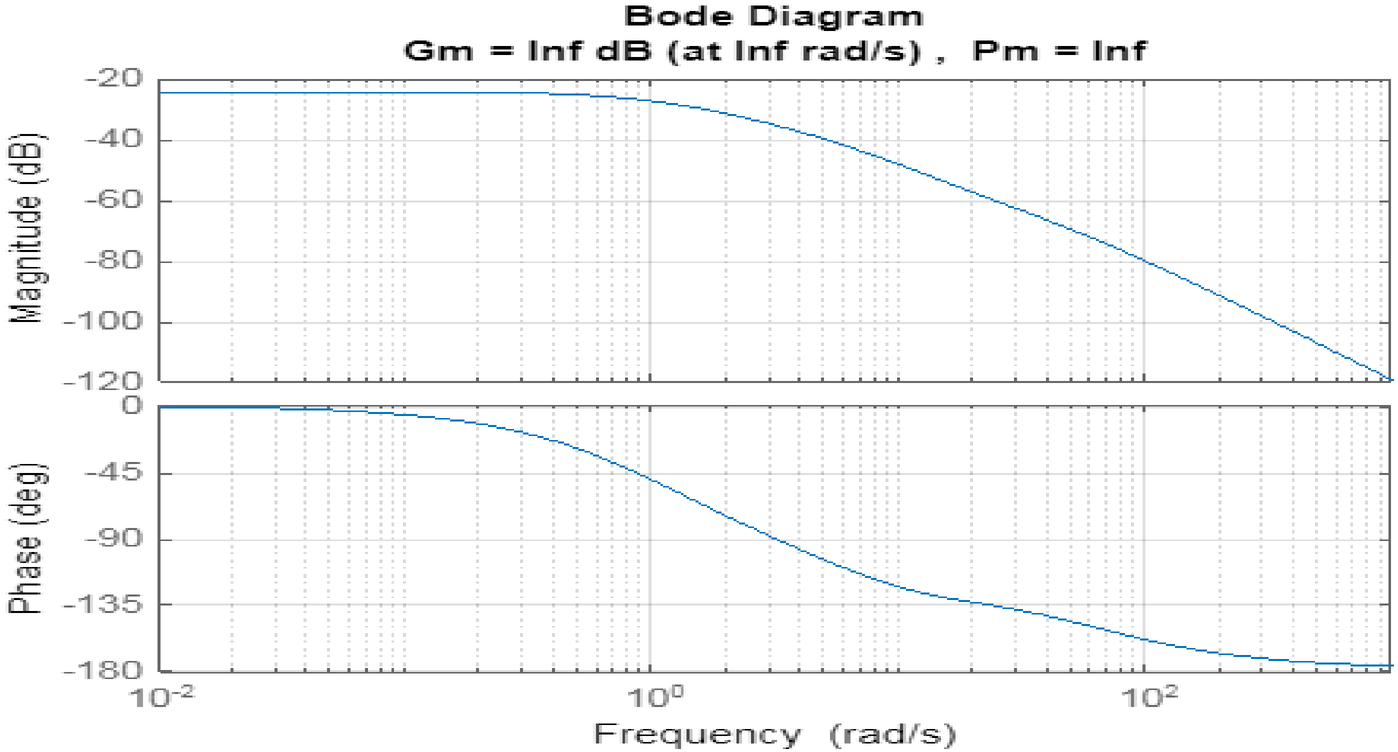




Bode Diagram
Gm = Inf dB (at Inf rad/s) , Pm = Inf
Magnitude (dB)
-20
-40
-60
-80
-100
Phase (deg)
0
-45
-90
-135
-180
10^-2
10^0
10^2
Frequency (rad/s)

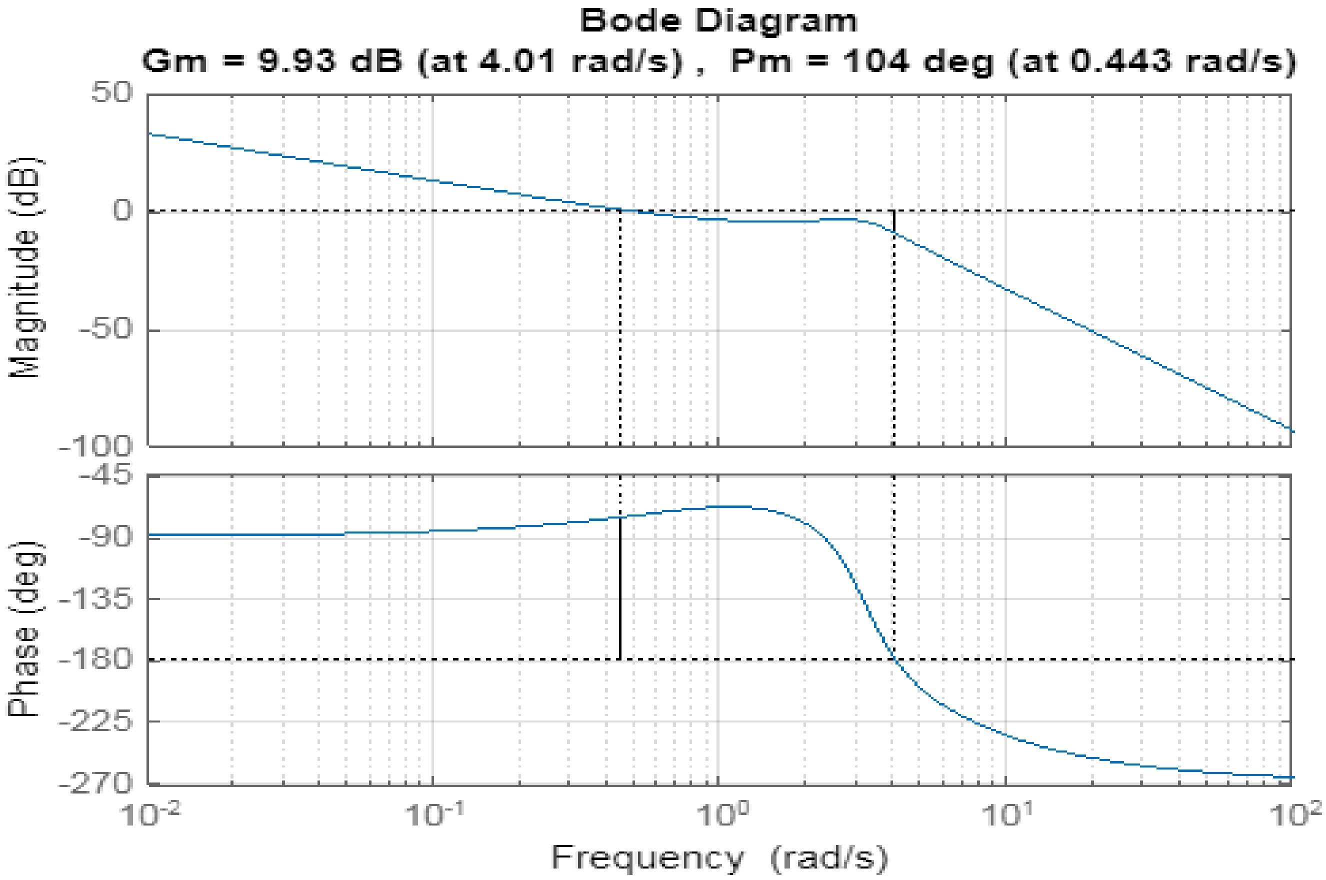
Problem 6

Draw the Bode log-magnitude and phase plots for the system $\dfrac{(s+20)}{(s+1)(s+7)(s+50)}$



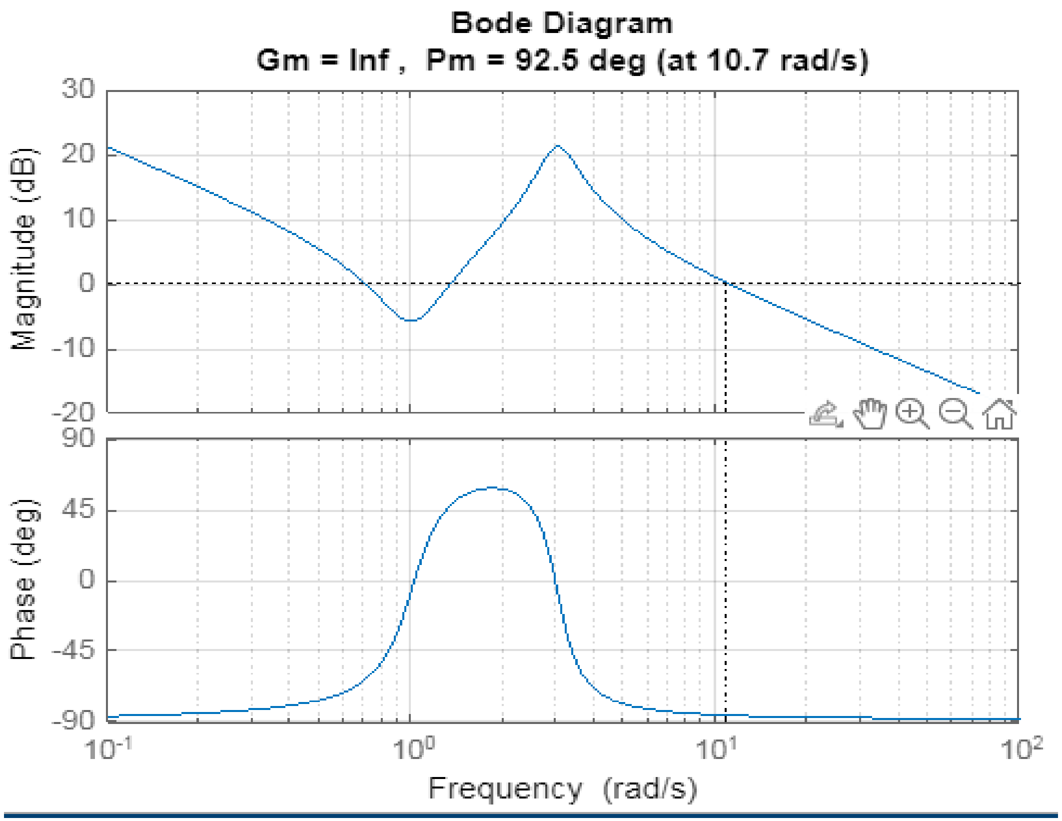
Problem 7

Draw the Bode log-magnitude and phase plots for the system $\frac{20(s+1)}{s(s+5)(s^2+2s+10)}$



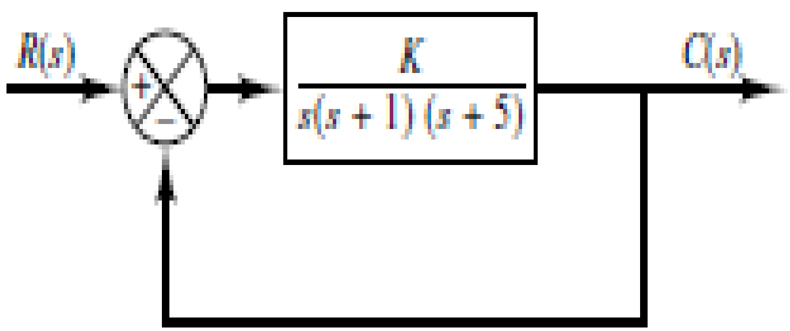
Problem 8

The open loop transfer function of a unity feedback system is given by $\frac{10(s^2+0.4s+1)}{s(s^2+0.8s+9)}$. Draw the Bode plot

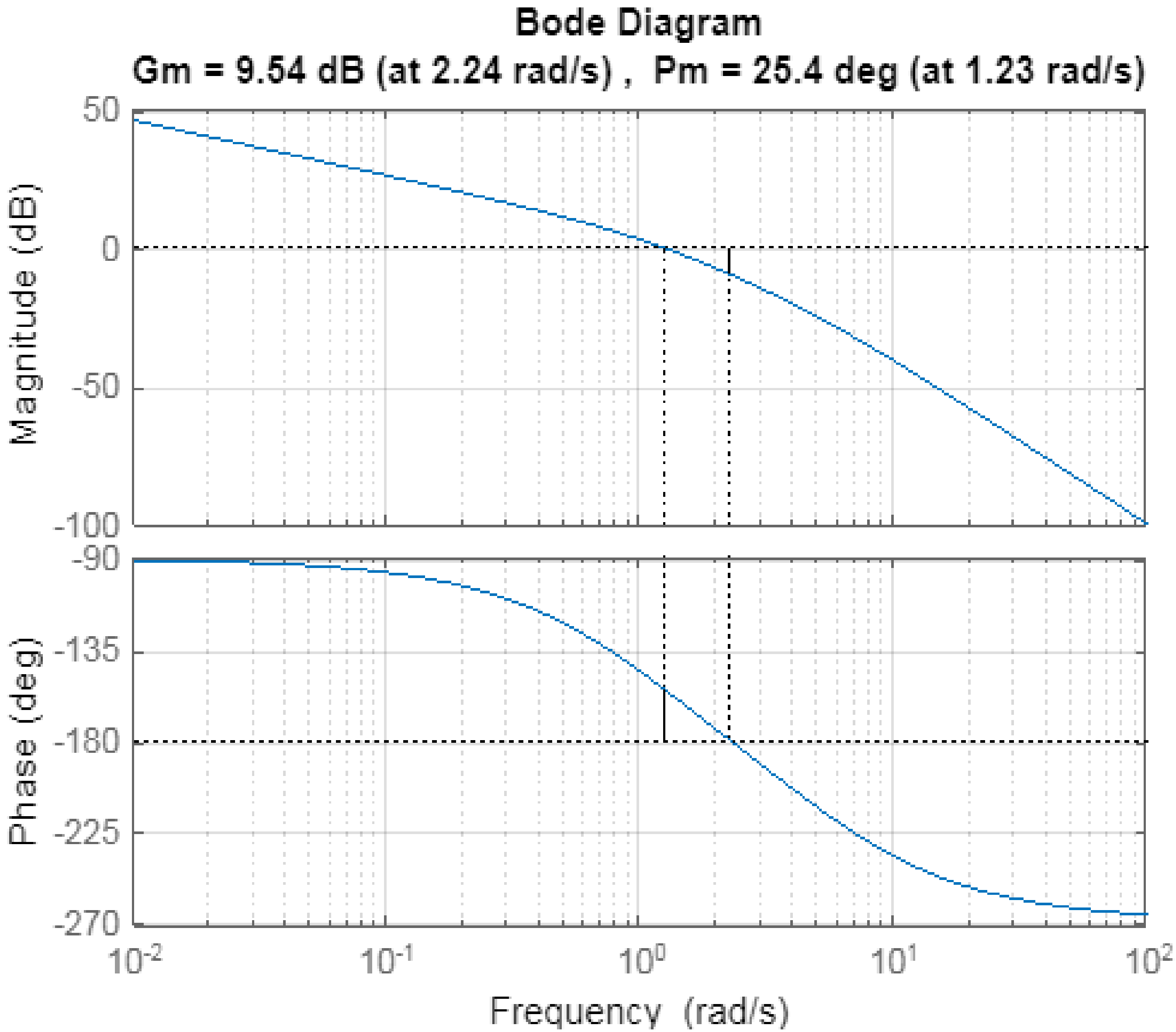


Problem 9

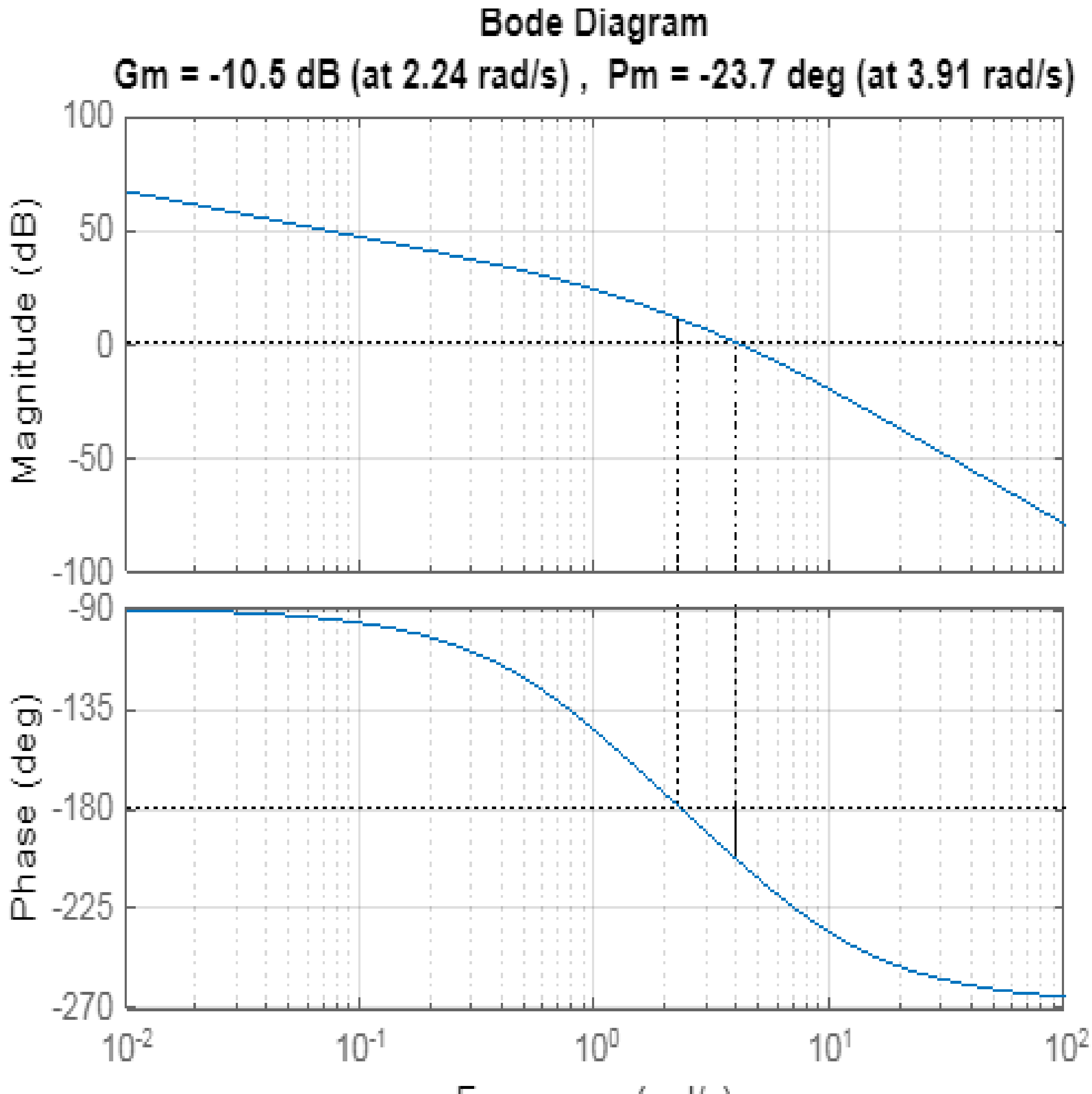
Obtain the phase and gain margins of the system shown in Figure for the two cases where K=10 and K=100



K=10


Bode Diagram
Gm = 9.54 dB (at 2.24 rad/s) , Pm = 25.4 deg (at 1.23 rad/s)
50
0
-50
-100
Magnitude (dB)
-90
-135
-180
-225
-270
Phase (deg)
10-2
10-1
100
101
102
Frequency (rad/s)

K=100


Bode Diagram
Gm = -10.5 dB (at 2.24 rad/s) , Pm = -23.7 deg (at 3.91 rad/s)
Magnitude (dB)
100
50
0
-50
-100
Phase (deg)
-90
-135
-180
-225
-270
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}

Problem

Draw the approximate (Asymptotic) Bode plot for the open loop transfer function G(s)= $\frac{(s+3)}{s(s+1)(s+2)}$

$$G(s) = \frac{(s+3)}{s(s+1)(s+2)} = \frac{3(\frac{s}{3}+1)}{s(s+1)2(\frac{s}{2}+1)} = \frac{1.5(0.33s+1)}{s(s+1)(0.5s+1)}$$

S=jω

$$G(j\omega) = \frac{1.5(0.33j\omega+1)}{j\omega(j\omega+1)(0.5j\omega+1)}$$

Magnitude

$$|G(j\omega)| = \frac{|1.5| * |(0.33j\omega+1)|}{|j\omega| * ||(j\omega+1)| * |(0.5j\omega+1)|}$$

$$M_{db} = \frac{20\log|1.5| + 20\log|(0.33j\omega+1)|}{20\log|j\omega| + 20\log|(j\omega+1)| + 20\log|(0.5j\omega+1)|}$$

$$= 20\log|1.5| + 20\log|(0.33j\omega + 1)| - 20\log|j\omega|$$

$$- 20\log|(j\omega + 1)| - 20\log|(0.5j\omega + 1)|$$

$$= M_1 + M_2 + M_3 + M_4 + M_5$$

$\mathbf{M_1 = 20\log 1.5 = 3.5\ dB}$ **constant**

$\mathbf{M_2 = 20\log|(0.33j\omega + 1)|}$ **zero at -3 rad/sec**

Corner frequency 3 rad/sec

magnitude is 0 dB upto CF

above CF , slope= 20 dB/dec

$\mathbf{M3 = -\ 20log\left| j\omega \right|}$ Pole at origin

Slope = -20dB/dec

$\mathbf{M_4\ = -\ 20log\left| (j\omega + 1) \right|}$ Pole at -1

Corner frequency = 1 rad/sec

Magnitude = 0 dB upto 1 rad/sec,

Above CF, slope = -20 dB/dec

$\mathbf{M_5 = -20log\left| (0.5j\omega + 1) \right|}$ Pole at -2

Corner frequency = 2 rad/sec

Magnitude = 0 dB upto 2 rad/sec,

Above CF, slope = -20 dB/dec

Phase

$\text{Ø} = \angle 1.5 + \angle(1+0.33j\omega) - \angle j\omega - \angle(1+j\omega) - \angle(1+0.5j\omega)$

$= 0 + \angle(1+0.33j\omega) - 90 - \angle(1+j\omega) - \angle(1+0.5j\omega)$

$= \text{Ø}_1 + \text{Ø}_2 + \text{Ø}_3 + \text{Ø}_4 + \text{Ø}_5$

$\text{Ø}_1 = \angle 1.5 = 0$ degrees (constant)

$\text{Ø}_2 = \angle(1+0.33j\omega)$ (zero at 3rad/sec)

Corner frequency= 3 rad/sec

At CF, Ø = 45 degrees, above C.F = 90 degrees

$\text{Ø}_3 = -\angle j\omega$ (pole at origin)

$= -90$ degrees

$\text{Ø}_4 = -\angle(1+j\omega)$ (pole at 1 rad/sec)

C.F=1 rad/sec

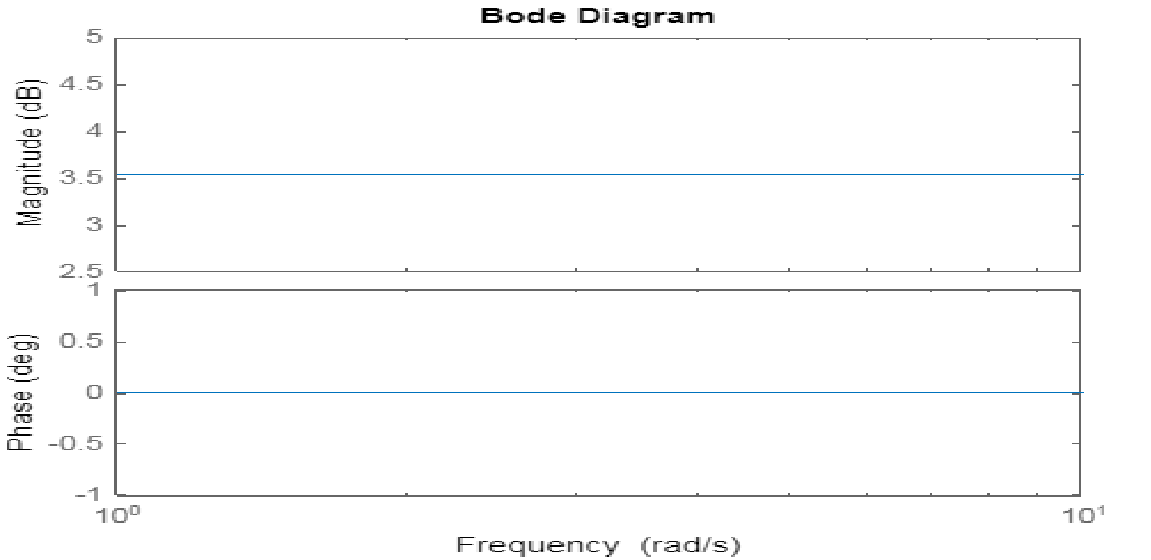
At CF, Ø = -45 degrees, above C.F = -90degrees

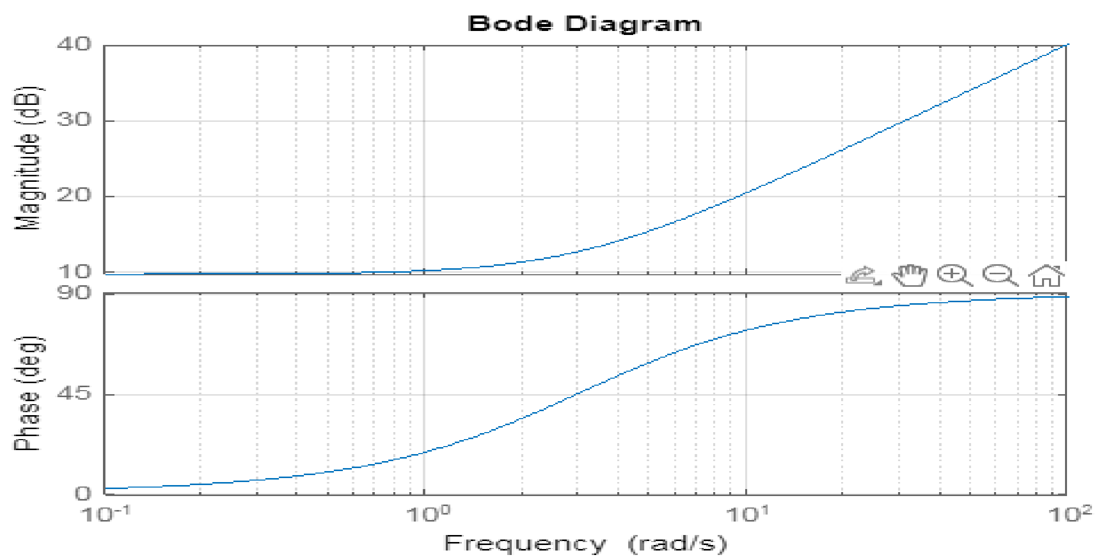
$\text{Ø}_5 = -\angle(1+0.5j\omega)$ (pole at 2 rad/sec)

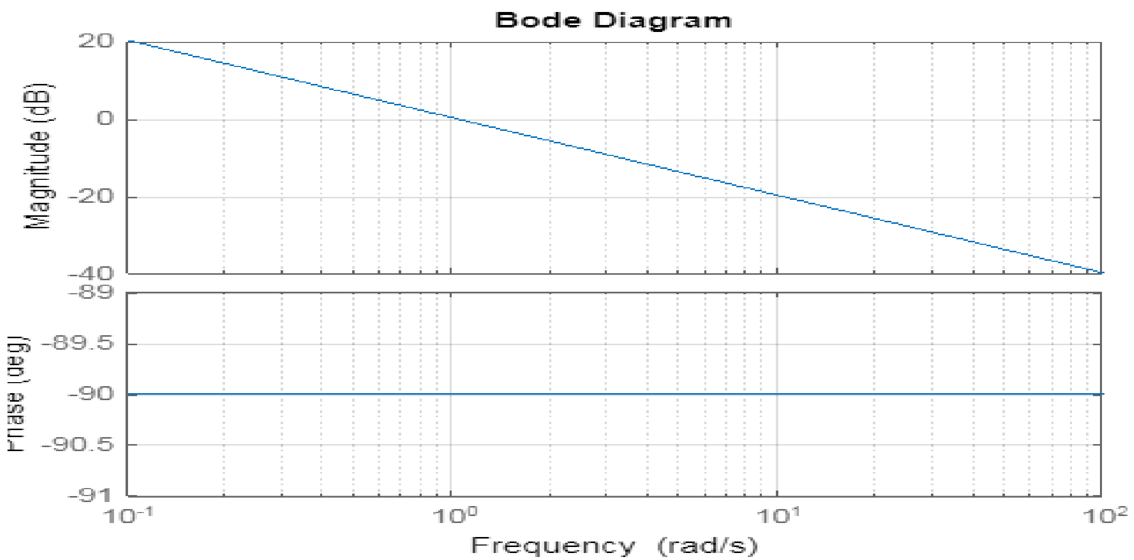
C.F = 2 rad/sec

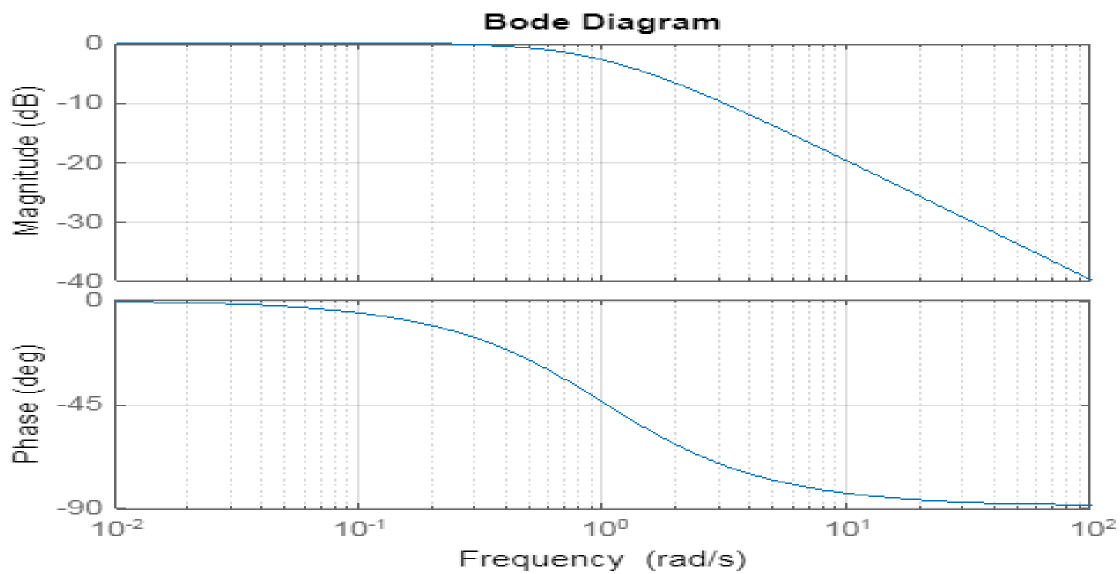
At CF, Ø = -45 degrees, above C.F = -90 degrees

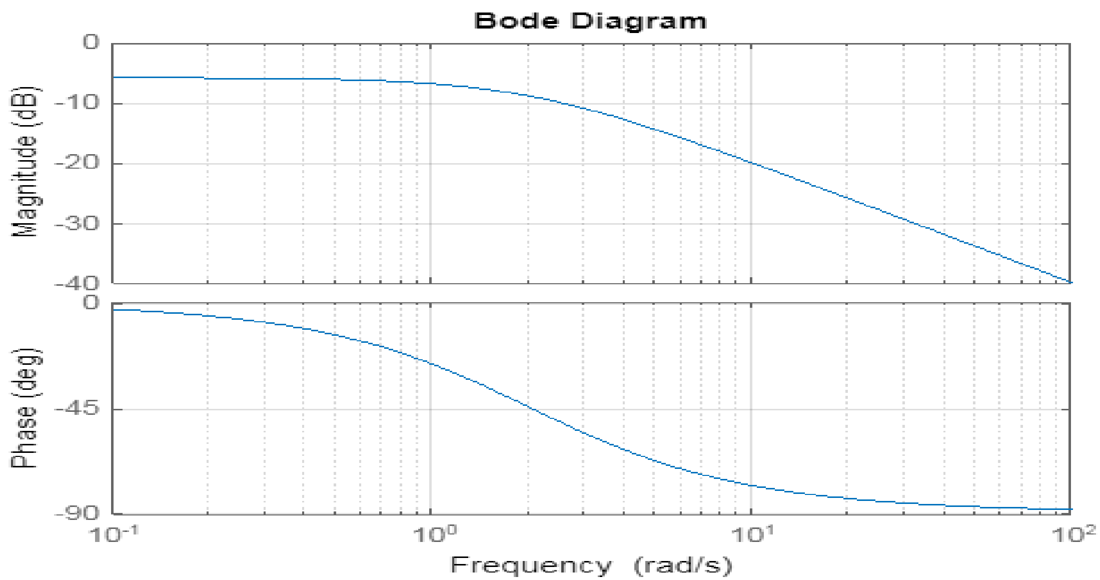
Factor	Corner Frequency (rad/sec)	Magnitude Characteristics	Phase Characteristics
1.5	-	Straight line of 3.5 db	zero
$\left(\frac{1}{j\omega}\right)$	-	Straight line of constant slope = -20db/decade passing through zero db at $\omega = 1$	Constant = -90^o
$\left(\frac{1}{1+j\,\omega}\right)$	ω_1=1	Straight line of 0 db for $\omega<\omega_1$, straight line of slope = -20db/decade for $\omega>\omega_1$	Varies from 0 to -90^o , at ω_1= -45^o
$\left(\frac{1}{1+j\,0.5\omega}\right)$	ω_2 = 2	Straight line of 0 db for $\omega<\omega_2$, straight line of slope = - 20db/decade for $\omega>\omega_2$	Varies from 0 to -90^o, at ω_2= -45^o
$(1+j0.33\omega)$	ω_3 = 3	Straight line of 0 db for $\omega<\omega_3$, straight line of slope = 20db/decade for $\omega>\omega_3$	Varies from 0 to 90^o, at ω_3= 90^o

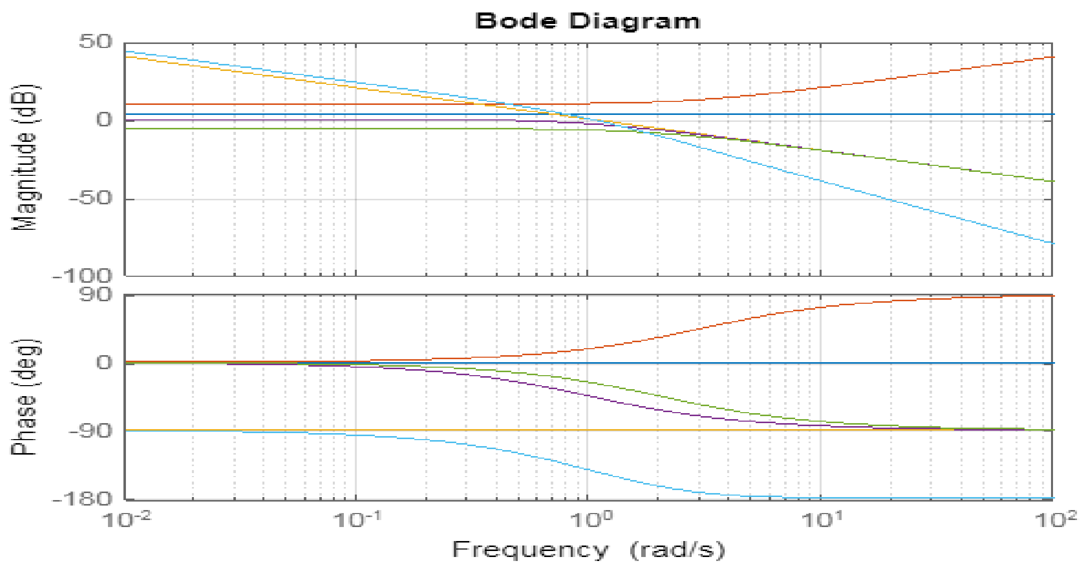

Bode Diagram
Magnitude (dB)
5
4.5
4
3.5
3
2.5
Phase (deg)
1
0.5
0
-0.5
-1
10^0
10^1
Frequency (rad/s)

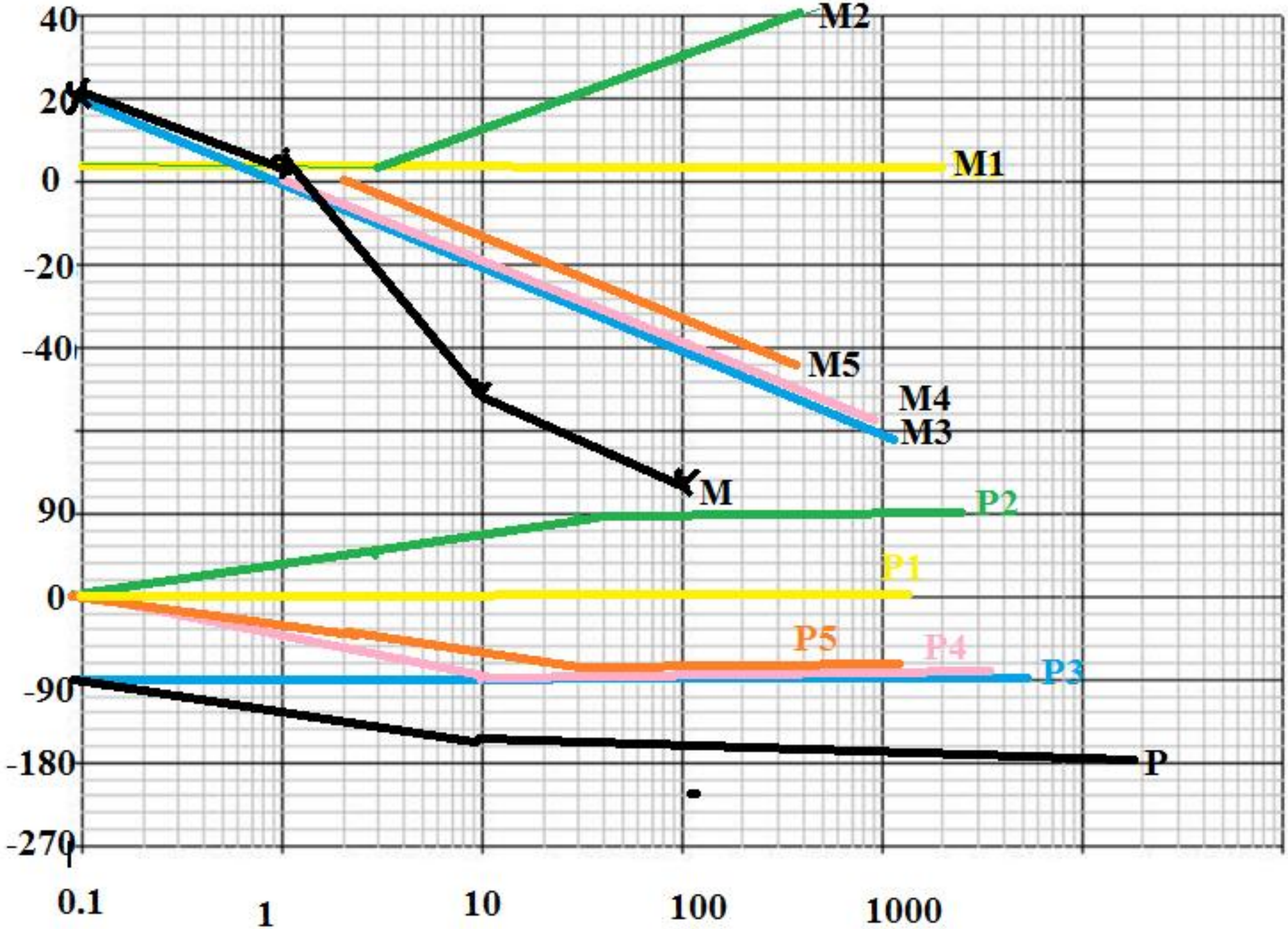

Bode Diagram
Magnitude (dB)
40
30
20
10
Phase (deg)
90
45
0
10^-1
10^0
10^1
10^2
Frequency (rad/s)


Bode Diagram
Magnitude (dB)
20
0
-20
-40
Phase (deg)
-89
-89.5
-90
-90.5
-91
10-1
100
101
102
Frequency (rad/s)

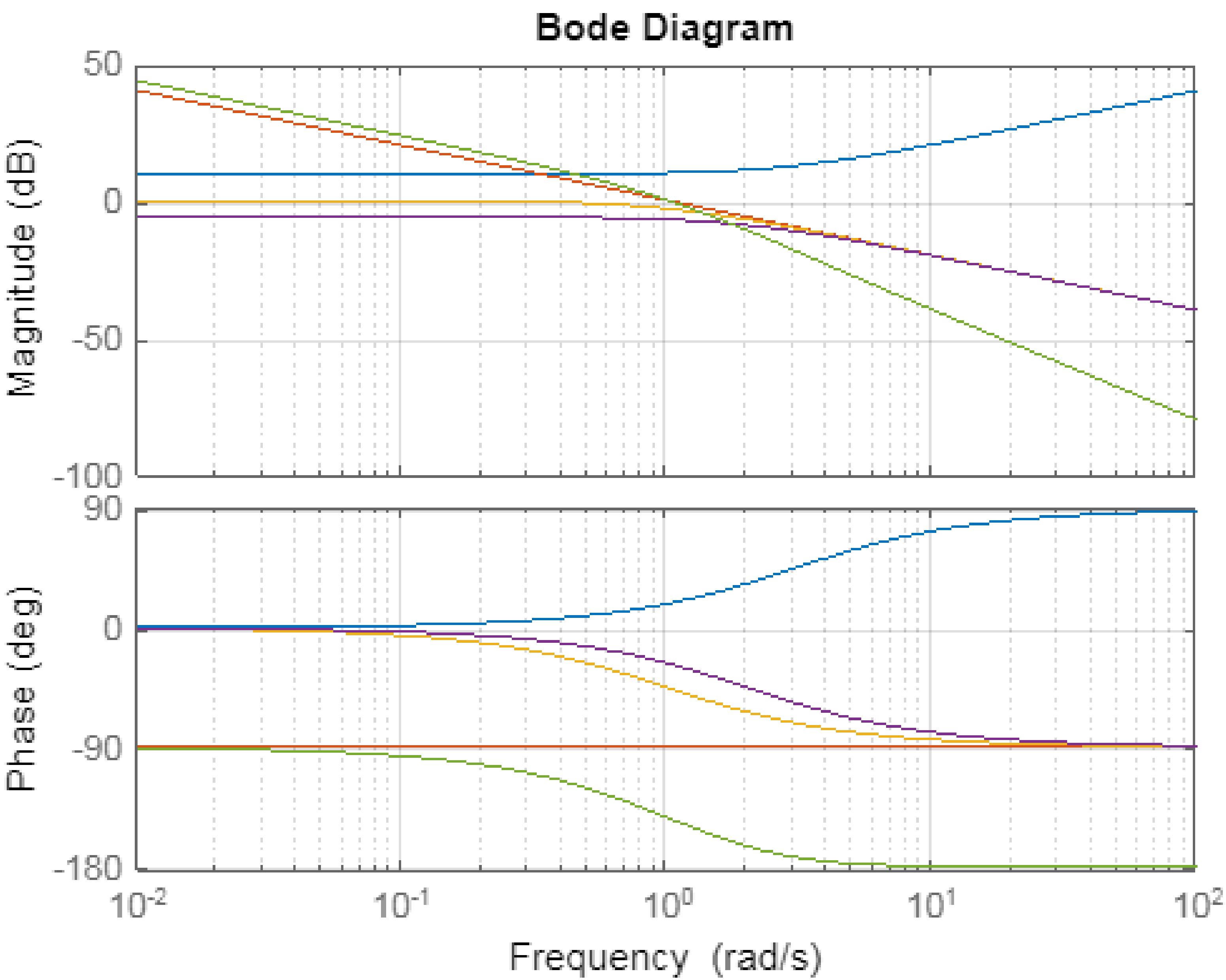

Bode Diagram
Magnitude (dB)
0
-10
-20
-30
-40
Phase (deg)
0
-45
-90
10-2
10-1
100
101
102
Frequency (rad/s)

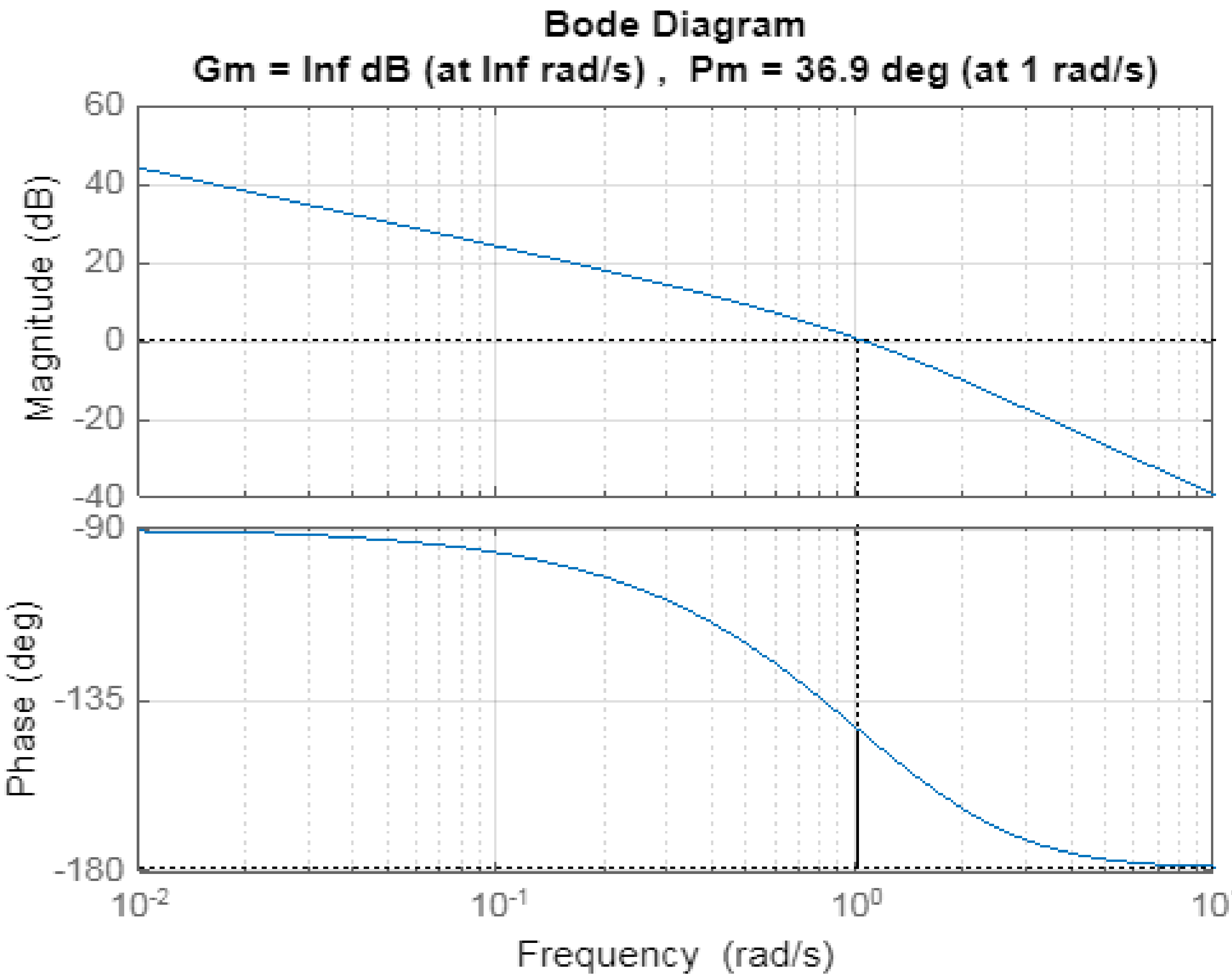

Bode Diagram
Magnitude (dB)
0
-10
-20
-30
-40
Phase (deg)
0
-45
-90
10-1
100
101
102
Frequency (rad/s)


Bode Diagram
Magnitude (dB)
50
0
-50
-100
Phase (deg)
90
0
-90
-180
10-2
10-1
100
101
102
Frequency (rad/s)



40
20
0
-20
-40
90
0
-90
-180
-270
0.1
1
10
100
1000
M2
M1
M5
M4
M3
M
P2
P1
P5
P4
P3
P


Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/s)
50
0
-50
-100
90
0
-90
-180
10-2
10-1
100
101
102


Bode Diagram
Gm = Inf dB (at Inf rad/s) , Pm = 36.9 deg (at 1 rad/s)
Magnitude (dB)
60
40
20
0
-20
-40
Phase (deg)
-90
-135
-180
10^{-2}
10^{-1}
10^{0}
10^{1}
Frequency (rad/s)

Problem

The open loop transfer function of a unity feedback system is given by $\frac{64(s+2)}{s(s+0.5)(s^2+3.2s+64)}$. Draw the approximate Bode plot

$$G(s)=\frac{64(s+2)}{s(s+0.5)(s^2+3.2s+64)} = \frac{64*2\,(\frac{s}{2}+1)}{s*0.5\left(\frac{s}{0.5}+1\right)*64*(\frac{s^2}{64}+\frac{3.2s}{64}+1)}$$

$$=\frac{4(0.5s+1)}{s(2s+1)(\frac{s^2}{64}+\frac{3.2}{8*8}\,s+1)} = \frac{4(0.5s+1)}{s(2s+1)(\frac{s^2}{64}+\frac{0.4}{8}\,s+1)}$$

$$S=j\omega$$

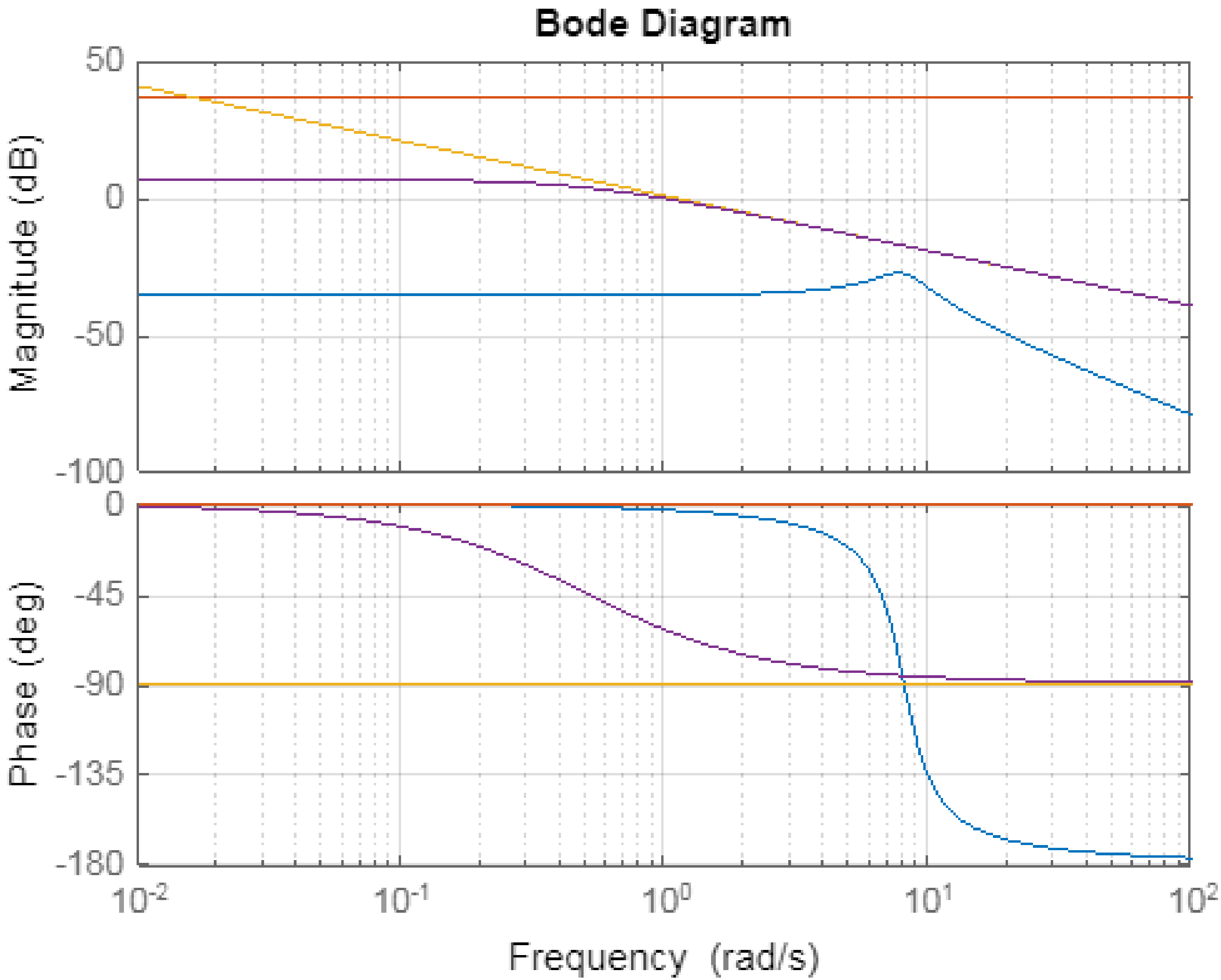
$$G(j\omega) = \frac{4(0.5j\omega+1)}{j\omega(2j\omega+1)((\frac{j\omega}{8})^2+j0.4(\frac{\omega}{8})+1)} = \frac{4(0.5j\omega+1)}{j\omega(2j\omega+1)(-(\frac{\omega}{8})^2+j0.4(\frac{\omega}{8})+1)}$$

Factors

The factors of this transfer function in order of their increasing frequencies are

1. Constant gain , k=4
2. Pole at origin, $1/j\omega$
3. Pole at s=-0.5; corner frequency $\omega_1 = 0.5$
4. zero at s=-2; corner frequency $\omega_2 = 2$
5. Pair of complex conjugate poles with $\zeta = 0.2$, $\omega_n = 8$; corner frequency $\omega_3 = 8$

Factor	Corner Frequency (rad/sec)	Magnitude Characteristics	Phase Characteristics
4	-	Straight line of 12 db	zero
$\left(\frac{1}{j\omega}\right)$	-	Straight line of constant slope = -20db/decade passing through zero db at $\omega = 1$	Constant = -90^o
$\left(\frac{1}{1+j\,2\omega}\right)$	ω_1=0.5	Straight line of 0 db for $\omega<\omega_1$, straight line of slope =-20db/decade for $\omega>\omega_1$	Varies from 0 to -90^o , at ω_1= -45^o
$1 + j0.5\omega$	ω_2 = 2	Straight line of 0 db for $\omega<\omega_2$, straight line of slope =20db/decade for $\omega>\omega_2$	Varies from 0 to 90^o, at ω_2= 45^o
$\frac{1}{1 + j0.4(\frac{\omega}{8}) - \left(\frac{\omega}{8}\right)^2}$	ω_3 = 8 ζ= 0.2	Straight line of 0 db for $\omega<\omega_3$, straight line of slope =-40db/decade for $\omega>\omega_3$	Varies from 0 to – 180^o, at ω_3= -90^o



Bode Diagram
Magnitude (dB)
50
0
-50
-100
Phase (deg)
0
-45
-90
-135
-180
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/s)

Problem

The system has an open loop transfer function $G(s) = \frac{10k}{s(1+0.05s)(1+0.1s)}$. Find the gain k such that

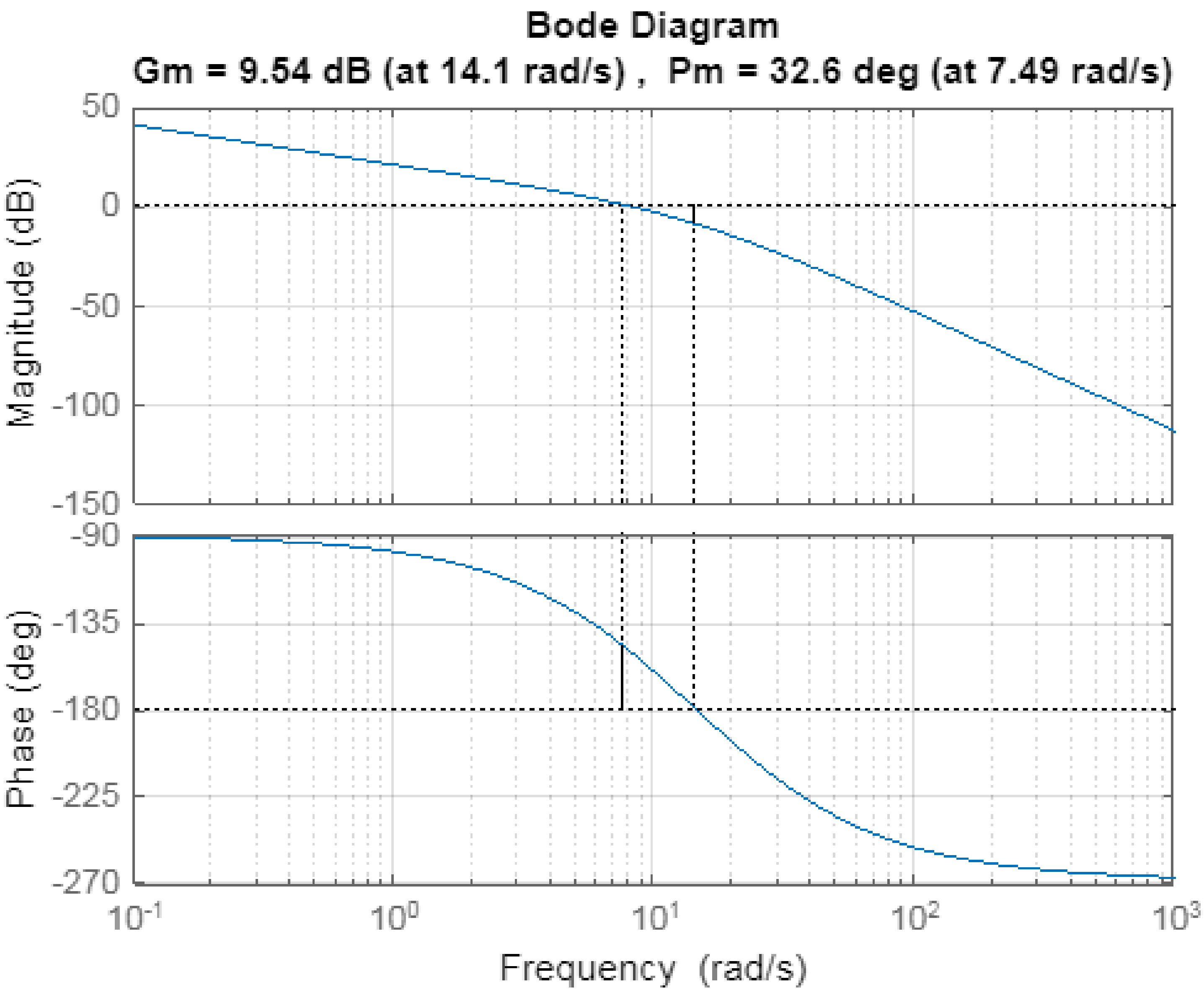
(i) GM = 20dB

(ii) PM = 10^o

Assume K=1, Draw the Bode plot

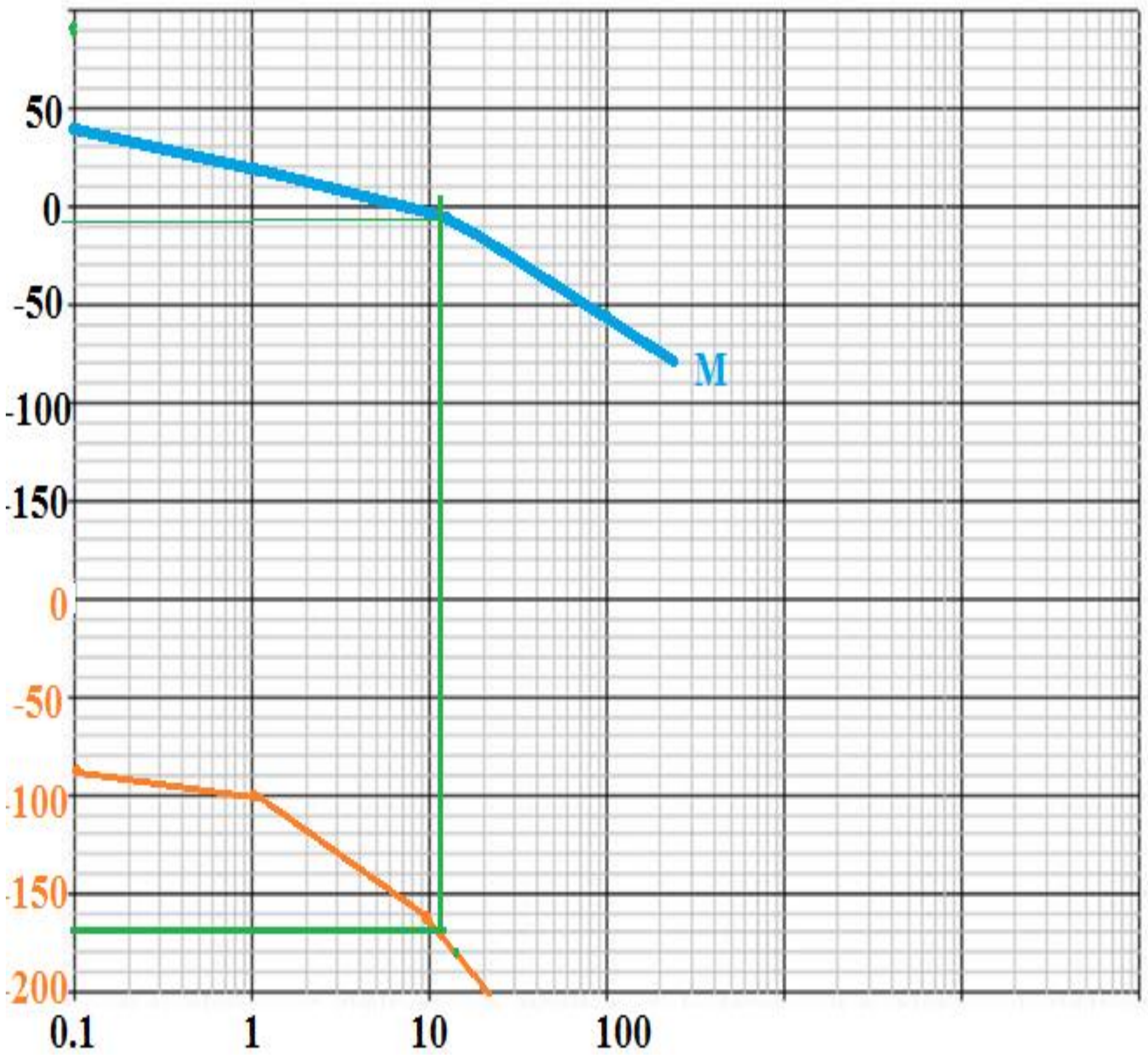
$$M_{db} = 20\log 10 - 20\log \omega - 20\log \sqrt{(1+(0.05\omega)^2)} - 20\log \sqrt{(1+(0.1\omega)^2)}$$

$$\emptyset = 0 - 90 - \tan^{-1}(0.05\omega) - \tan^{-1}(0.1\omega)$$



Bode Diagram
Gm = 9.54 dB (at 14.1 rad/s) , Pm = 32.6 deg (at 7.49 rad/s)
Magnitude (dB)
50
0
-50
-100
-150
Phase (deg)
-90
-135
-180
-225
-270
10-1
100
101
102
103
Frequency (rad/s)

ω	M	Ø
0.1	40	-91
1	20	-99
10	-4	-162
20	-16	-198
100	-54	-253



For k=1, ω_g= 7.49 rad/sec

ω_p= 14.1 rad/sec

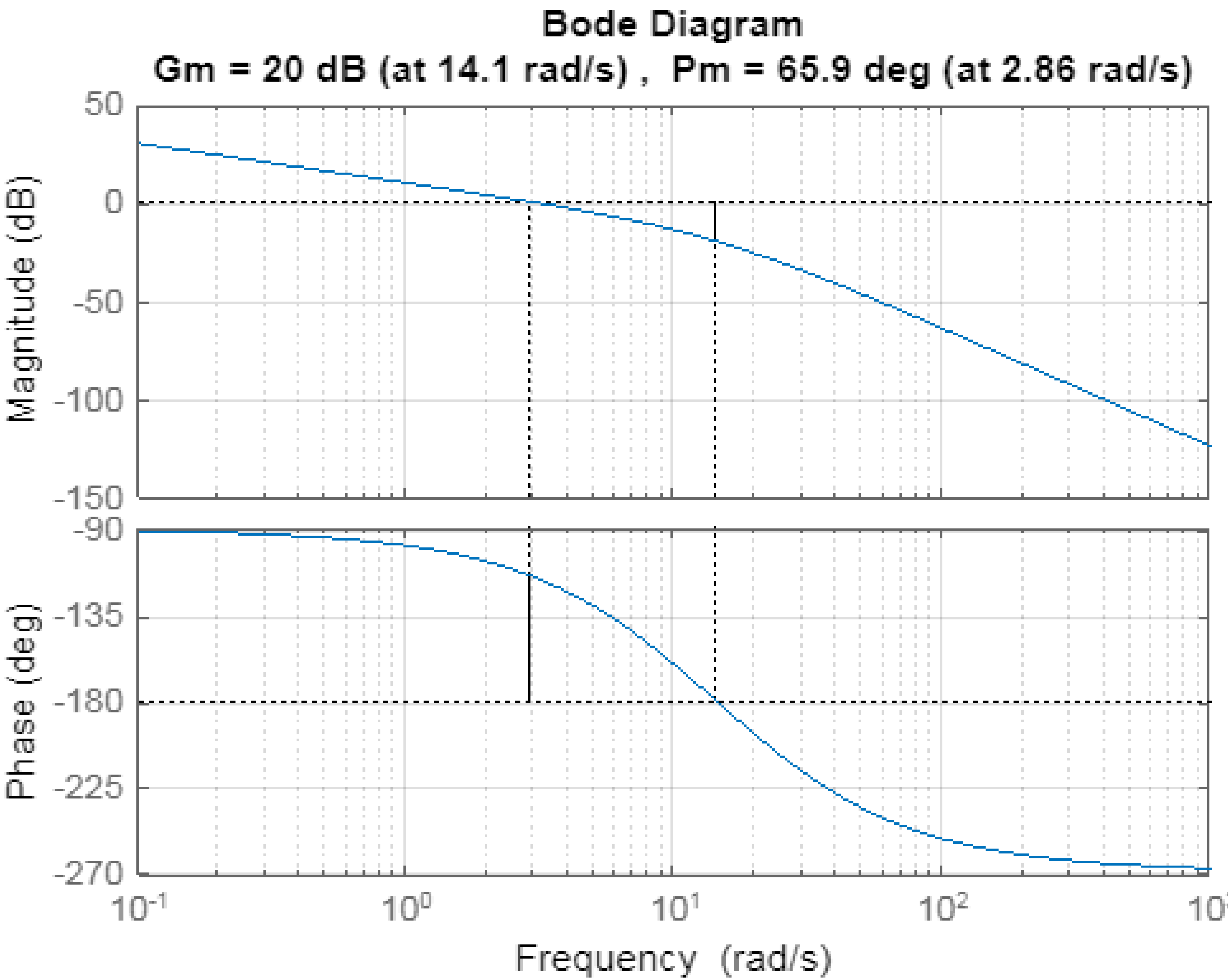
GM = 9.54 dB

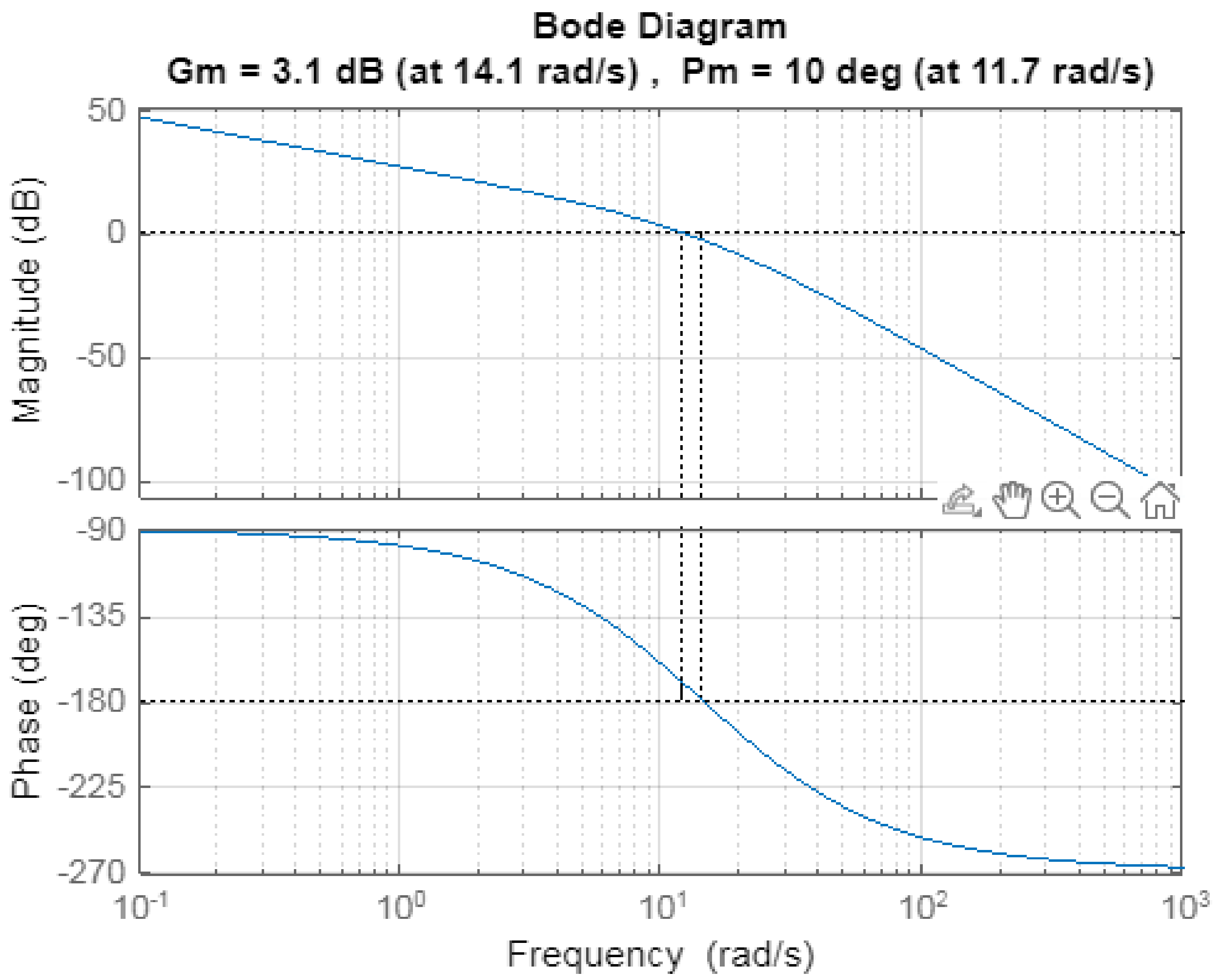
PM = 32.6^o

(i) To get GM= 20 dB ,the plot is shifted downwards by (20-9.54) = 10.46 dB.

The system gain is decreased by -10.46 dB or multiplied by a factor 0.32

(ii) To get PM= 10^o ,the plot is shifted upwards by 7 dB. The system gain is changed by 7 dB or multiplied by a factor 2.2

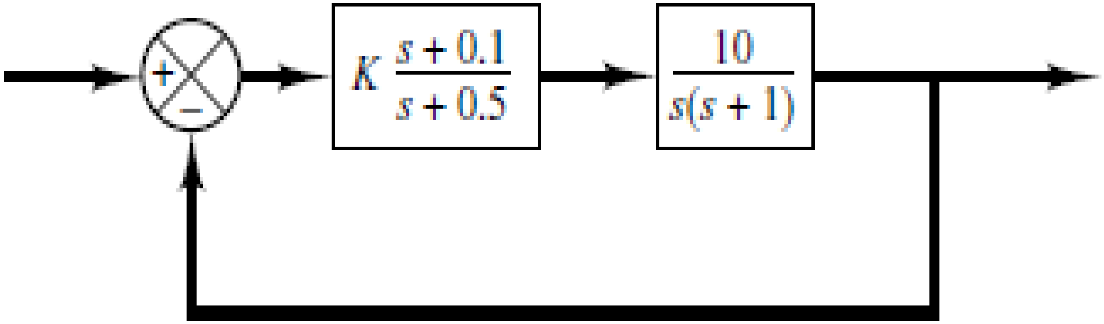

Bode Diagram
Gm = 20 dB (at 14.1 rad/s) , Pm = 65.9 deg (at 2.86 rad/s)
Magnitude (dB)
50
0
-50
-100
-150
Phase (deg)
-90
-135
-180
-225
-270
10-1
100
101
102
103
Frequency (rad/s)



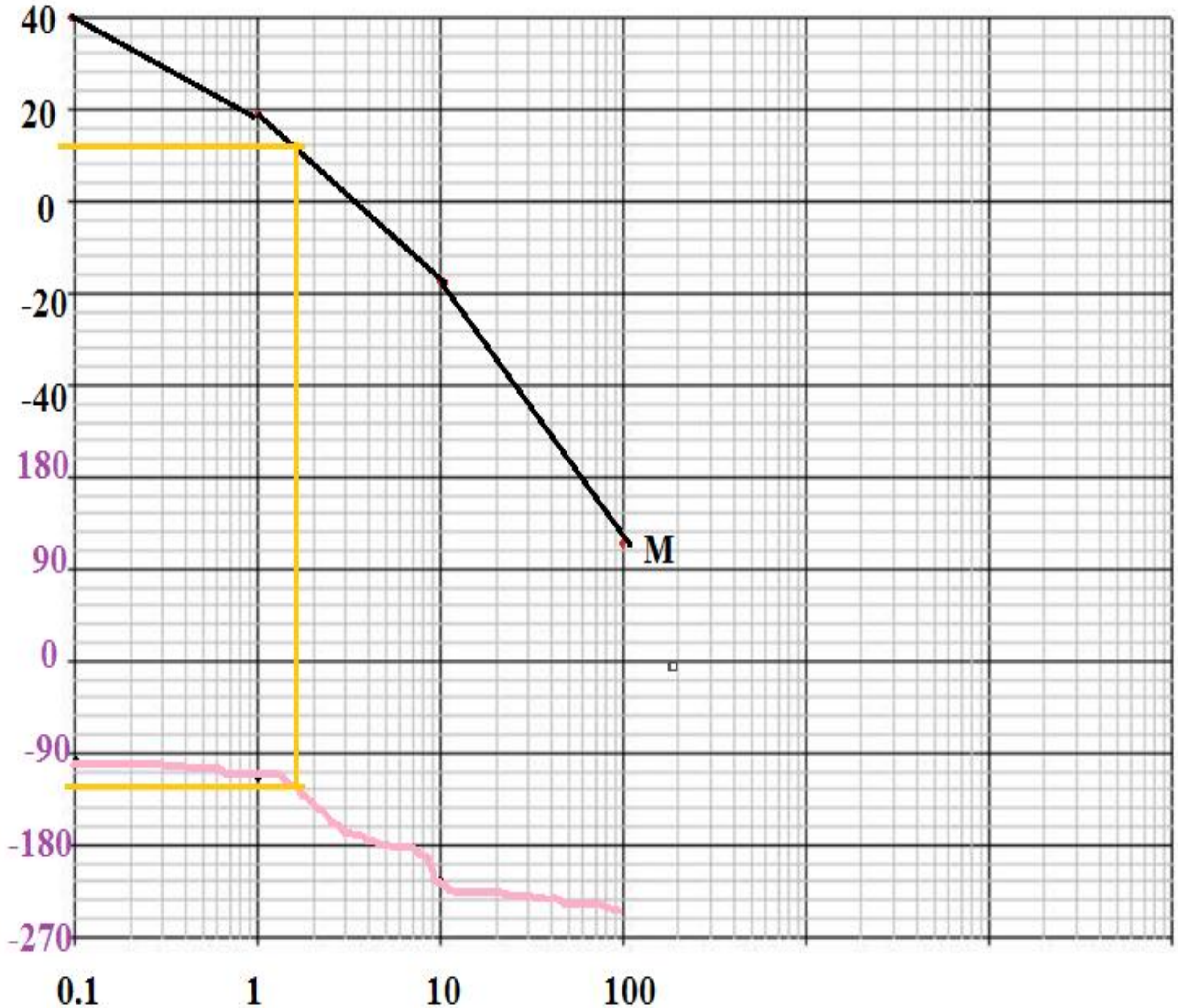
Bode Diagram
Gm = 3.1 dB (at 14.1 rad/s) , Pm = 10 deg (at 11.7 rad/s)
Magnitude (dB)
50
0
-50
-100
Phase (deg)
-90
-135
-180
-225
-270
10-1
100
101
102
103
Frequency (rad/s)

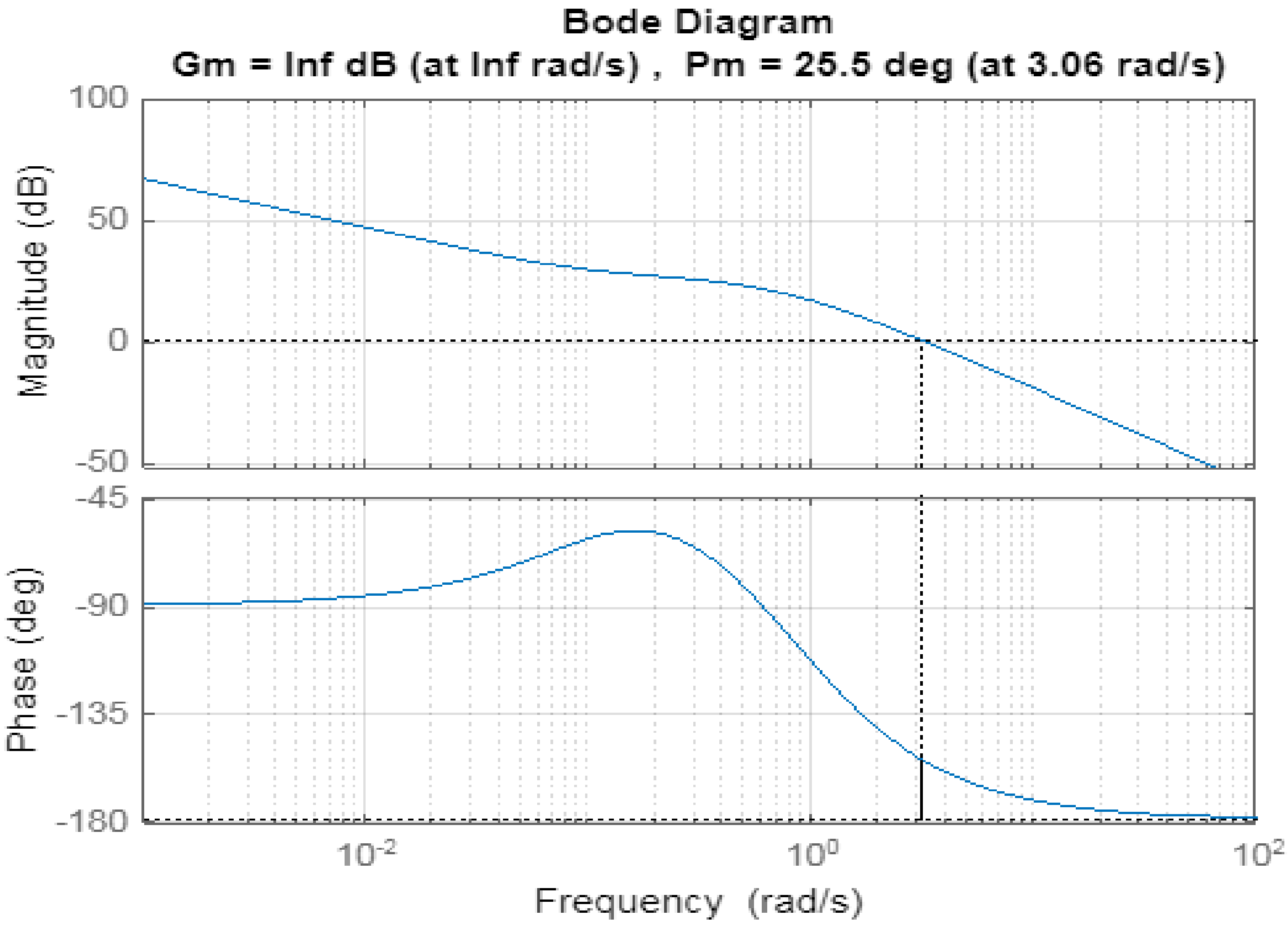
Problem

Consider the system shown in Figure. Draw a Bode diagram of the open-loop transfer function, and determine the value of the gain K such that the phase margin is 50°. What is the gain margin of this system with this gain K?

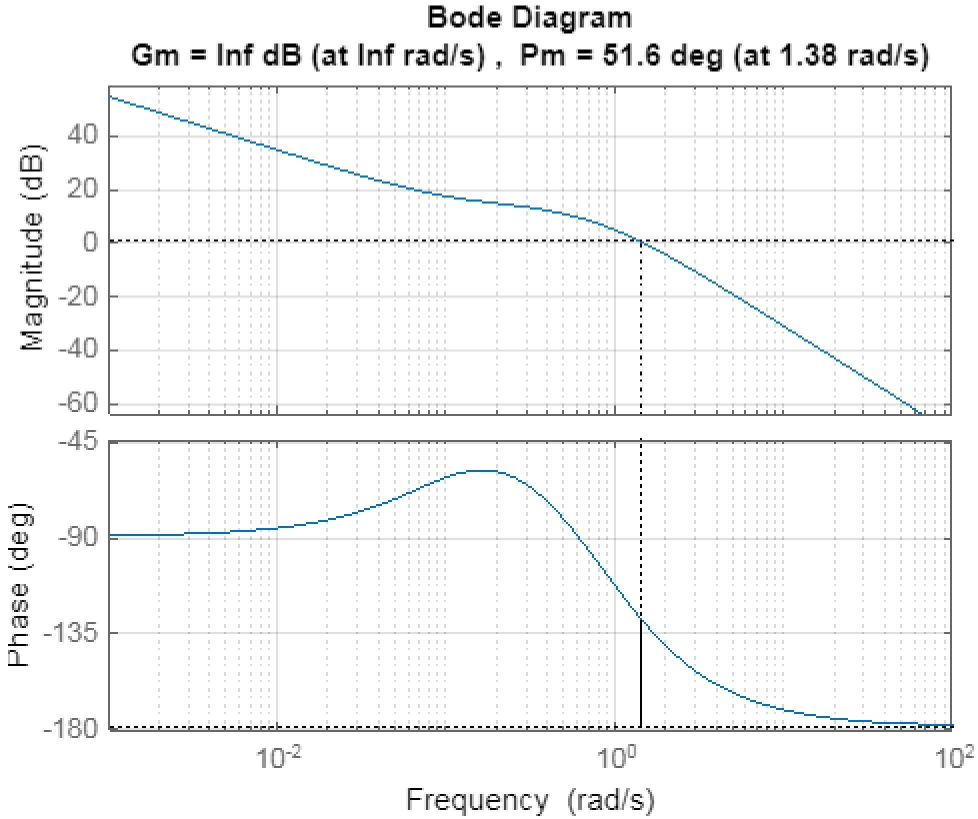


ω	M (db)	Ø (deg)
0.1	40	-93
1	19	-122.3
2	11	-146
10	-17.16	-213.7
100	-74	-263

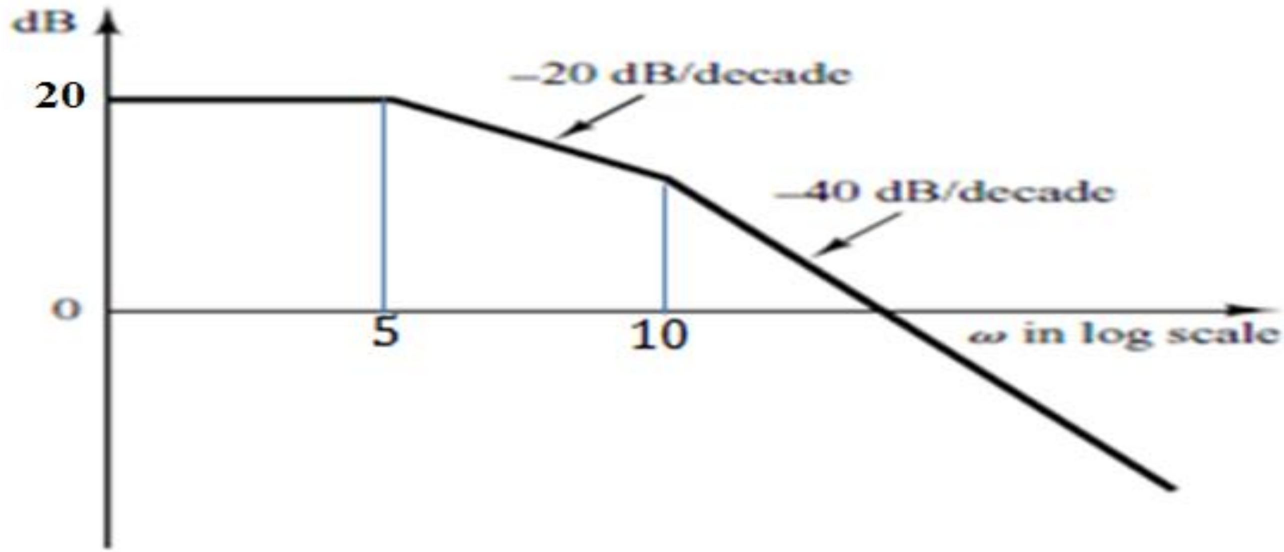



Bode Diagram
Gm = Inf dB (at Inf rad/s) , Pm = 25.5 deg (at 3.06 rad/s)
Magnitude (dB)
100
50
0
-50
Phase (deg)
-45
-90
-135
-180
10-2
100
102
Frequency (rad/s)

To get PM= 50^o , the plot is shifted upwards by 8 dB. The system gain is changed by 8 dB or increased by a factor 2.5



Transfer Function from Bode Plot



$$G(s) = \frac{k}{\left(1+\frac{s}{5}\right)\left(1+\frac{s}{10}\right)} = \frac{k}{\left(\frac{5+s}{5}\right)\left(\frac{10+s}{10}\right)} = \frac{k*50}{(5+s)(10+s)}$$

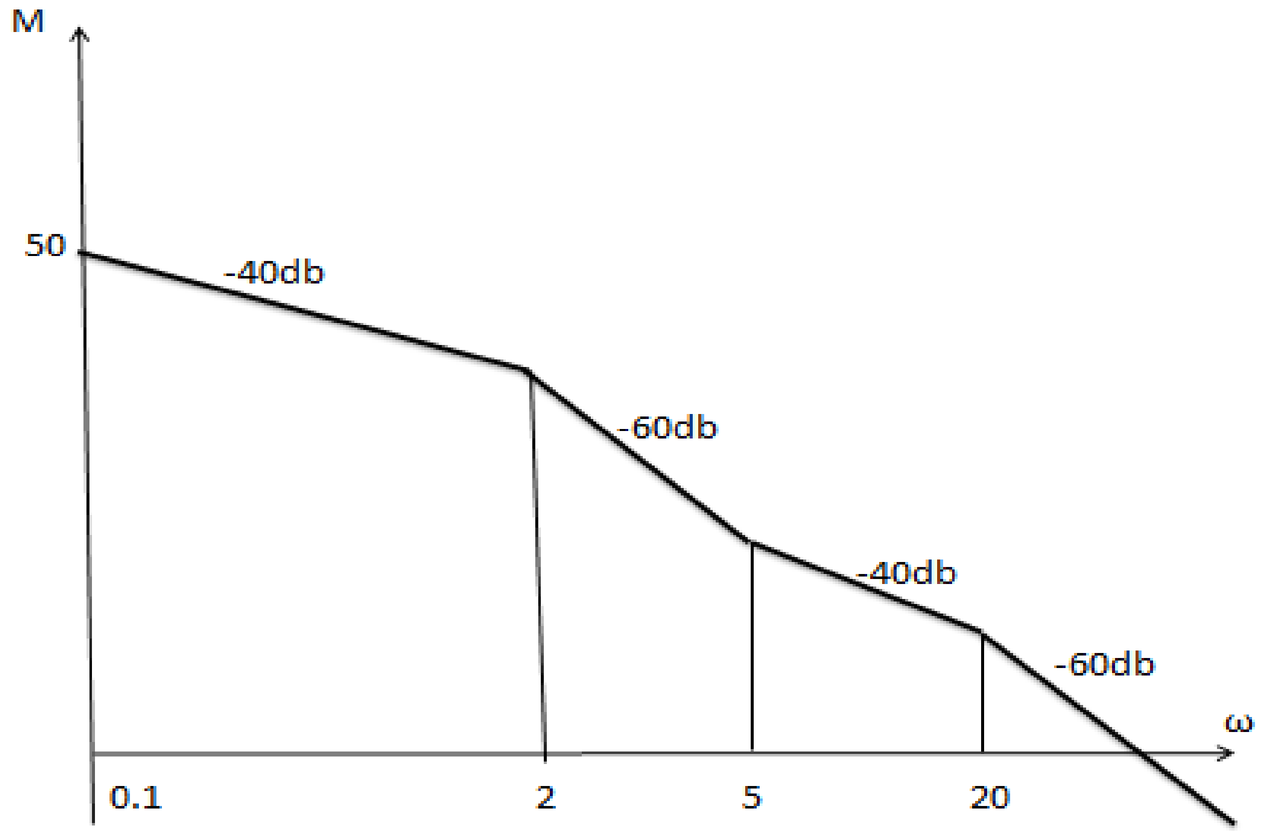
20 log k =20

k=1

$$G(s) = \frac{50}{(5+s)(10+s)}$$

Problem

Find the transfer function for the given Bode plot



$$G(s) = \frac{k(1+\frac{s}{5})}{s^2\left(1+\frac{s}{2}\right)(1+\frac{s}{20})}$$

At $\omega = 0.1$, M= 50db

$$20 \log(\frac{k}{\omega^2})=50$$

$$20\log k + 20 \log(\frac{1}{\omega^2}) = 50$$

$$20\log k - 40\log \omega = 50$$

$$20\log k - 40\log (0.1) = 50$$

$$20\log k + 40 = 50$$

$$20\log k= 10 , K = 10^{10/20} = 3.16$$

$$G(s) = \frac{3.16(\frac{5+s}{5})}{s^2\left(\frac{2+s}{2}\right)(\frac{20+s}{20})} = \frac{25.28(s+5)}{s^2(2+s)(20+s)}$$

References

1. Control Engineering by Nagrath & Gopal, New Age International Publishers
2. Engineering control systems - Norman S. Nise, John WILEY & sons , fifth Edition
3. Modern control Engineering-Ogata, Prentice Hall
4. Automatic Control Systems- B.C Kuo, John Wiley and Sons